ELEMENTS

OF THE

PRECISION OF MEASUREMENTS

AND

GRAPHICAL METHODS

BY

H. M. GOODWIN, Ph.D.
PROFESSOR OF PHYSICS
MASSACHUSETTS INSTITUTE OF TECHNOLOGY

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PREFACE.

In its present form the "Elements of the Precision of Measurements and Graphical Methods" represents the ground covered in a brief course which has been given for a number of years at the Massachusetts Institute of Technology to all students in connection with their work in the Physical Laboratory. The author has been induced to amplify the printed "Notes" on this subject and give them a wider circulation in response to repeated requests to use them elsewhere. Although prepared primarily to meet the needs of his own classes, it is hoped, in the present form, they may prove useful in other technical schools and colleges where quantitative work forms a part of the curriculum, and also to engineers whose work involves experimental testing. In many laboratories far too little weight is attached to the discussion of the magnitude and effect of sources of error on a result. This has been forced upon the writer's attention as the result of personal interviews with hundreds of graduate students entering the Institute, who apply for excuse from laboratory work. It is the exceptional student who has any conception how to figure out the precision of a final computed result from the precision of his individual measurements, and this is true even though his laboratory note-book shows his work to have been carefully and creditably performed. It is the author's firm conviction that one of the most valuable and enduring benefits of physical laboratory training to a student of Science or Engineering is the acquisition of the proper view-point with which to approach an investigation, be it either purely scientific or technical; that is, the ability to recognize the essentials of a problem at the outset, so as to economize both time and labor in its solution. Although the exercise of judgment, based upon the personal experience of the investigator, is essential to the "best solution" of any experimental problem, still it is desirable to direct the student's attention to precision methods at an early stage of his laboratory work. Experience has shown that this may be satisfactorily done as soon as he has had a little practice in exact measurements and can handle the elements of Differential Calculus. At the Institute the course is given at the middle of the sophomore year, after the student has performed some six or eight experiments on fundamental measurements in Mechanics. Continued application of the principles is then made in subsequent laboratory work throughout the junior and senior years, and a precision discussion is regarded as

an important feature of the final thesis. It has been the writer's experience that students have little trouble in understanding the general principles involved, but meet with considerable difficulty in applying these principles to concrete problems. For this reason the subject is most satisfactorily taught to small sections by recitations based on the text and the solution of numerous problems selected from the book and from the current laboratory work. A close correlation of class-room and laboratory work is indeed highly desirable, and in the Rogers Laboratory of Physics it is the practice to require with each laboratory report a precision discussion of the data or a solution of some precision problem related to the experiment. The laboratory manuals have been written with this in view.

The method of treatment has been kept as brief as possible. A full discussion of the subject, with proofs based on the Theory of Probability and the Method of Least Squares, would so enlarge the work as to defeat its end. Proofs of the few theorems and formulæ which the student is asked to assume may be found in any good treatise on Least Squares. An excellent treatment is that given in Bartlett's "Method of Least Squares." A more exhaustive treatment of Precision Methods may be found in Holman's "Precision of Measurements."

A chapter on the solution of illustrative problems has been added to assist students who find it necessary to work up the subject by themselves. The collection of problems has been compiled from recent examination papers. The chapter on Graphical Methods contains specific directions for constructing graphs, and general directions for obtaining therefrom the functional relationship between two variables. For engineering students, as well as physicists, the method of logarithmic plotting will be found of wide application. In the Appendix several tables, of assistance in precision computations, have been added.

In conclusion the author desires to express his indebtedness for many suggestions to his colleagues who have so ably assisted him in the instruction of this subject in recent years, and in particular to Professor William J. Drisko, whose experience in teaching this and related subjects has been most helpful.

H. M. Goodwin.

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PRECISION OF MEASUREMENTS.

Classification of Physical Measurements.—All physical measurements may be classed as direct or indirect according as the measurement gives the desired result directly, or as the result is obtained by combining the results of several measurements by means of some formula. Examples of the first class are the measurement of a length by means of a scale, the mass of a body by means of an equal arm balance, and the electrical resistance of a wire by the direct method of substitution. Examples of indirect measurements are the determination of g, the acceleration due to gravity, by means of a pendulum, involving the measurement of the length and time of vibration of the pendulum, the determination of the index of refraction of a substance from measurements of the angle and the minimum deviation of a prism by means of a spectrometer, and the determination of the specific heat of a substance by the method of mixtures in which the results of the measurement of a number of temperatures and weights are combined. The great majority of problems arising in practice come under the second class.

Reliability of a Result.—In order that the result of any measurement, whether direct or indirect, may be of any scientific or technical value, it is necessary to have some numerical estimate or measure of its reliability. The importance of such a measure cannot be overestimated. The result of a test, of a study of an instrument or method, or of the determination of a constant, may be rendered almost worthless, unless the investigator is able to state the degree of reliance which can be placed upon it. This phase of an investigation should be kept constantly in mind in all laboratory work. The student's ability to intelligently dis-

cuss the reliability of his data is regarded as of no less importance than his ability to perform accurate work.

By the *precision* or *precision measure* of a result, denoted for brevity by *p.m.*, will be always understood the best numerical measure of its reliability which can be obtained after all known sources of error have been eliminated or corrected for. How this may be computed will be explained below. By the *accuracy* of a result should, strictly speaking, be understood the degree of concordance between it and the true value of the quantity measured. Since, however, the latter is usually unknown, it is seldom that we can obtain a numerical measure of the absolute accuracy of a measurement. We must in most cases be content with an estimated or computed precision measure. The terms "accuracy" and "precision" are often carelessly used indiscriminately.

The precision measure of a direct measurement is of no less importance than of an indirect measurement. As the precision of the latter depends primarily on the precision of the separate components from which it is computed, the method of determining a numerical estimate of the reliability of a series of direct observations will first be considered.

Classification of Errors.—When any quantity is measured to the full precision of which the instrument or method employed is capable, it will in general be found that the results of repeated measurements do not exactly agree. This is true not only of results obtained by different observers using different instruments and methods, but also when the measurements are made by the same observer under similar conditions. The cause of these discrepancies lies in various sources of error to which all experimental data are subject. These may be grouped conveniently in two classes,—determinate and indeterminate errors.

Determinate Errors.—Determinate errors are, as their name indicates, of such a nature that their value can be determined and their effect on the result thereby eliminated. They may be classified as follows:—

a. Instrumental Errors.—These may arise from poor con-

struction or faulty adjustment of an instrument, as, for example, a defect in a micrometer screw, faulty graduations of scales and circles, eccentricity of circles, unequal balance arms, etc.

b. Personal Errors.—These may arise from characteristic peculiarities of individual observers, as, for example, the tendency to always record the occurrence of an event too soon or too late. This frequently happens in recording transit observations in which the "personal equation" of the observer becomes an important factor.

c. Errors of Method or Theoretical Errors.—These may arise from using an instrument under conditions for which its graduations are not standard.

The following illustrations will make clearer the nature of the above sources of error. Suppose that the arms of a chemical balance are slightly unequal in length. All weighings made with such a balance will be in error due to this cause (if the balance be used in the ordinary way), by an amount depending on the inequality in the length of the arms. Repeated weighings of the same substance on the same balance by the same method will, however, give no indication of the presence of this source of error. The repeated independent weighings may indeed check among themselves to the full sensitiveness of the balance, and yet the result may be in error, due to the constant instrumental error, by a very large amount. The presence of such an error would only be detected by comparing the results of the weight of the same body obtained on different balances or by different methods of weighing, for the probability of the same instrumental error occurring to the same extent in different instruments is very small.

Again, suppose a length is measured by means of a graduated scale at 20° C., while the scale is standard at some other temperature, say 0° C. Repeated measurements with such a scale by the same method and under the same conditions would probably show a very close agreement among themselves, and give no clew to the presence of any constant

error. The result would, however, be too small, since the value of the units of the scale would all be too large, due to the expansion of the scale from 0° to 20°. The error thus introduced by using the scale under conditions other than those for which it is standard is, however, determinate in its nature, since a knowledge of the coefficient of expansion of the scale and of the temperature at which it is standard, and also at which it is used, furnishes all necessary data for reducing the observed result to the value it would have had, had the scale been standard at the time of the measurement. The concordance of a series of observations taken under similar conditions is, therefore, no criterion of the absence of constant errors even of very large amount.

To detect and eliminate such errors, it is necessary to compare the results of measurements of the quantity by different methods, different apparatus, and, if possible, different observers, and to average such independent results by a special method described later; for the probability of the same source of error being present under such variable conditions is very small. An interesting illustration of the presence of a constant error escaping detection is to be found in the original determination of the ohm by the British Association Committee. The excellent agreement of the observations among themselves lead to the conclusion that the result possessed a high degree of reliability. Later determinations by independent methods and observers gave values which differed from the B. A. value by over one per cent., an amount far in excess of the precision with which the B. A. determination had been carried out. Attention was thus called to the probable presence of some constant error which further investigation verified.

Residual Errors.—After a result has been corrected as well as may be for all known sources of determinate errors, there may still remain in it small errors, the value of which cannot be determined, and which, therefore, fall into the second general class of errors,—indeterminate errors. Thus, if the instrumental error arising from inequality in the length of

balance arms be corrected by a determination of the ratio of the arms, this ratio will be known to only a certain degree of precision, and hence the corrected result of a weighing may still be in error by an amount depending on the precision with which the correction itself has been determined. Or, again, correcting for the expansion of a scale involves an experimental investigation of the coefficient of expansion of the material of which the scale is constructed, and this constant can be determined with only a certain degree of precision. A result corrected by means of this coefficient will, therefore, still be uncertain beyond a certain point due to the uncertainty in the value of the coefficient used. These small errors remaining, because of the impossibility of completely correcting for constant errors, are called residual errors: their numerical value and algebraic sign cannot be determined, but usually limiting values may be estimated and assigned to them. For this reason they are properly grouped and treated under the second general class of errors mentioned,—indeterminate errors.

Indeterminate Errors.—*Accidental; Residual.*—Experience shows that, when a measurement is repeated a number of times with the same instrument and by the same observer under apparently the same conditions, the results usually differ in the last place or sometimes last two places of figures. Thus in so simple a measurement as the determination of the distance between two lines with a scale graduated in millimeters, successive measurements will not agree to one-tenth millimeter if fractions of a millimeter are estimated by the eye. Errors which give rise to such variations which at one time cause a result to be too high and at another too low are due to causes over which the observer has no control, such as sudden temperature fluctuations which may give rise to unequal expansion of different parts of an apparatus, or to changes in refraction, barometric changes, shaking of the instrument due to mechanical jar or to the wind, etc.; and, more important still, to physiological causes arising from imperfections or fatigue of the eye or ear

of the observer. The magnitude and sign of errors arising from such causes have been shown, however, to follow a perfectly definite law,—namely, the law of chance. The nature of this law may be illustrated as follows. Suppose a thousand shots be fired at a target by a skilled marksman under conditions as nearly alike as possible. Experience shows that the shots will be distributed in a manner which at first sight seems entirely irregular, but which on more careful examination will be found to be approximately in conformity with a perfectly definite law. In an actual case obtained with a target ruled in horizontal sections by lines one foot apart, the centre line (corresponding to the bull's eye) being in the middle of one of these spaces, the following results were obtained:—

In space	*No. of shots*
$+5\frac{1}{2}$ to $+4\frac{1}{2}$	1
$+4\frac{1}{2}$ " $+3\frac{1}{2}$	4
$+3\frac{1}{2}$ " $+2\frac{1}{2}$	10
$+2\frac{1}{2}$ " $+1\frac{1}{2}$	89
$+1\frac{1}{2}$ " $+\frac{1}{2}$	190
$+\frac{1}{2}$ " $-\frac{1}{2}$	212
$-\frac{1}{2}$ " $-1\frac{1}{2}$	204
$-1\frac{1}{2}$ " $-2\frac{1}{2}$	193
$-2\frac{1}{2}$ " $-3\frac{1}{2}$	79
$-3\frac{1}{2}$ " $-4\frac{1}{2}$	16
$-4\frac{1}{2}$ " $-5\frac{1}{2}$	2

If a plot be made with the number of shots falling in the several sections as ordinates and the distance of the corresponding spaces from the central line as abscissæ, we obtain Figure 1. From this it appears that plus and minus deviations of the shots from the central line are about equally frequent, and that small deviations occur with much greater frequency than large ones. If the number of shots (corresponding to observations) be increased, the irregularities present in the curve will tend to smooth out, and it can be shown mathematically that in the limit the curve represent-

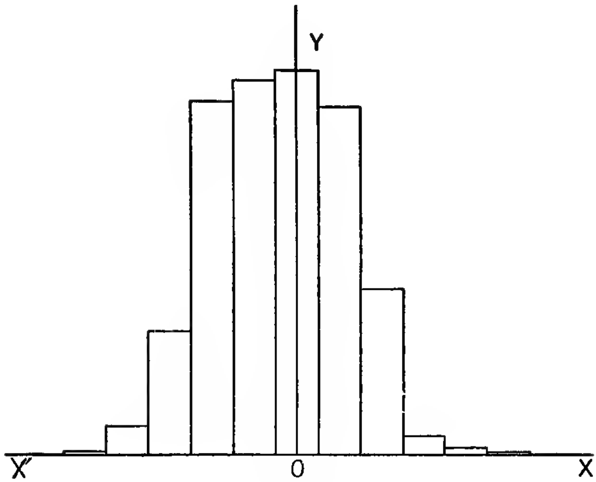


Fig. 1.

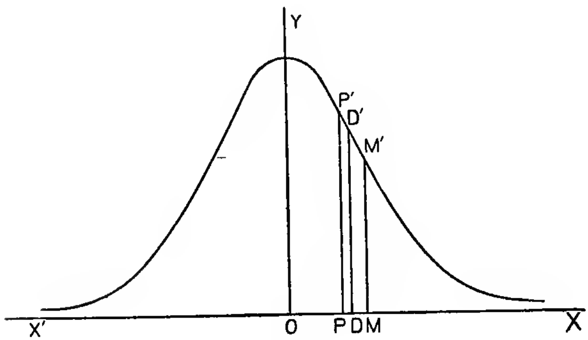


Fig. 2.

ing the law of chance takes the general form shown in Figure 2, the equation of which is

$$y = \frac{h}{\sqrt{\pi}} e^{-h^2x^2}.$$

Here y is the frequency of the occurrence of an error of the magnitude x, and h is a constant, the value of which depends on the character of the observations and which affords a measure of their precision. The curve represented by this

equation is called the Curve of Error. By inspection it is seen that:—

First.—Small errors occur more frequently than large ones (curve of error has a maximum for $x = o$);

Second.—Very large errors are unlikely to occur (curve is asymptotic to the axis of X);

Third.—Positive and negative errors of the same numerical magnitude are equally likely to occur (curve of error is symmetrical with respect to axis of Y).

Since accidental and residual errors of a series of observations follow the law of chance, they may be properly subjected to mathematical treatment based on this law. It must be remembered, however, that, since the law itself represents a limiting case, corresponding to an infinite number of observations, deductions from it apply to a finite number of observations only with a certain probability which becomes less the smaller the number of observations.

The Method of Least Squares.—As already pointed out, in the great majority of measurements the true value of the quantity is unknown and cannot be determined. Were it known, a measurement would be superfluous. All that we can hope to obtain from our experimental data is the most probable value of the quantity or quantities in question. In many cases this is a simple matter; but in others, where the number of observations is larger than the number of unknown quantities to be determined, the problem may become one of some difficulty. The branch of mathematics which treats of the general problem of the adjustment of errors of observation so that their effect upon the result is reduced to a minimum, and the best representative values of the desired quantities thus obtained, is called Least Squares, the name being derived from the criterion upon which the adjustment of the observations is based. This states that the most probable values of a series of related observations arc those for which the sum of the squares of the errors is a minimum. Certain deductions from the theory of Least Squares will be assumed as demonstrated in the course of

this work. For the proofs the student is referred to Bartlett's Method of Least Squares or other treatises on the subject. An illustration of the method as applied to the computation of the constants of an empirical equation is given under Graphical Methods.

The Arithmetical Mean.—Deviation Measures.—We will now consider the precision discussion of a series of direct measurements. Let $a_1, a_2, \ldots a_n$ be a series of observations on a quantity, all of which possess an equal degree of probability. Under these conditions the most probable value of the quantity is given by the arithmetical mean, m, of the series, *i.e.*

$$m = \frac{\Sigma a}{n}. \tag{1}$$

Since the true value of the quantity is unknown, the error of each of the observations and of the mean, m, cannot be determined. We can, however, obtain a numerical measure of the amount by which each observation differs from the mean value, and from this the *probable deviation* of the mean can be computed. The difference between the value of any observation of a series and the mean value of the series is called the *deviation of that observation from the mean.* It is to be distinguished from the absolute error of the observation, *i.e.*, the difference between the observed value and its true value, from which it may differ widely. Deviations as thus computed follow the same law as indeterminate errors, *i.e.*, the law of chance, and are subject, therefore, to the same mathematical treatment. They give a measure of the magnitude of the accidental error of a measurement, but evidently afford no indication of the presence or magnitude of any constant errors which may be present.

If the numerical deviations $d_1, d_2, d_3, \ldots d_n$, be computed for any series of observations as above, their algebraic sum will be zero, since the sum of the positive deviations is equal to the sum of the negative deviations. If, however, their arithmetical mean be computed, disregarding their sign, the

result will be a number which expresses how much on the average any single observation of the series taken at random is likely to differ (plus or minus) from the mean, m. This average value

$$a.d. = \frac{\Sigma d}{n} \qquad (2)$$

is called the *average deviation of a single observation,* and will be denoted by $a.d.$ In recording data as $a_1, a_2, \ldots a_n$, space should always be left for computing the deviations $d_1, d_2, \ldots d_n$, respectively, and their $a.d.$ as follows:—

$$\begin{array}{rcl} a_1 - m &=& d_1 \\ a_2 - m &=& d_2 \\ a_3 - m &=& d_3 \\ \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot \\ a_n - m &=& d_n \end{array}$$

$$m = \frac{\Sigma a}{n} \qquad a.d. = \frac{\Sigma d}{n}$$

Looked at from another point of view, an $a.d.$ is a numerical measure of the amount by which a new observation taken under the same conditions as before is likely to differ from the mean value, m. It gives a numerical measure of the reliability of any single observation of the series so far as accidental errors affecting the measurement are concerned.

Deviation of the Mean, A.D.—In general, however, it is the reliability of the mean that we desire to know rather than that of a single observation. As the mean has a higher degree of probability than any single observation from which it is computed, it must evidently have a smaller deviation than a single observation in proportion to its greater reliability. It can be shown that an arithmetical mean computed from n equally probable observations is $\sqrt{n}$ times as reliable as any one observation. Hence, if the deviation

measure of a single observation of a series is *a.d.*, the deviation measure of the mean of n such observation is only $\frac{1}{\sqrt{n}}$ as great; *i.e.*, the deviation of the mean, denoted by *A.D.*, is,

$$A.D. = \frac{a.d.}{\sqrt{n}}. \tag{3}$$

Thus, if the mean value of nine measurements of the distance between two lines is 1.3215 mm. and the average deviation of any one of the measurements is found to be $a.d. =$ 0.0033 mm., the mean will have a probable deviation not greater than $\frac{0.0033}{\sqrt{9}} = 0.0011$ mm. From this it will be seen that in general it does not pay to increase the number of observations beyond a certain limit, say nine or sixteen, as the time and labor involved soon become excessive, without a corresponding increase in the precision attained.

Fractional and Percentage Deviation Measures.— It is frequently convenient to express the reliability of a quantity as a fractional or as a percentage part of the quantity itself. Thus we have in very common use the two following deviation measures derived from the preceding:—

the fractional deviation of a single observation $= \frac{a.d.}{a}$;

the percentage deviation of a single observation $= 100\,\frac{a.d.}{a}$;

the fractional deviation of the mean $= \frac{A.D.}{m}$;

the percentage deviation of the mean $= 100\,\frac{A.D.}{m}$.

Since these measures are never computed to more than two significant figures, see page 23, a and m, being approximately the same, may be used indiscriminately in the computation.

Deviation Measure vs. Precision Measure.— A little consideration will make clear that all of the above *deviation* measures give a measure of the magnitude of errors of that

type which has been classed as *accidental.* A result may be seriously in error due to residual errors, and yet the observations show a good agreement among themselves, and their deviation measure be correspondingly small. If the magnitude of the residual errors can be estimated, we may compute the true *precision measure,* abbreviated *p.m.*, of the result as follows:—

Let the estimated magnitude of the residual errors be $r_1, r_2, \ldots r_n$. Let *d.m.* represent the value of the deviation measure of the accidental errors. This may be expressed as an average, fractional or percentage deviation, but, whichever is chosen, the residuals must be expressed in the same way. It can then be shown that the most probable measure of the reliability of the result will be given by the expression:—

$$\overline{p.m.}^2 = \overline{d.m.}^2 + r_1^2 + r_2^2 + \ldots + r_n^2, \tag{4}$$

or
$$p.m. = \sqrt{\overline{d.m.}^2 + r_1^2 + r_2^2 + \ldots + r_n^2}. \tag{4a}$$

Thus the precision measure of a result differs from its deviation measure in that it includes the effect of residual as well as of accidental errors. In a great many cases the value of the residuals is negligible compared with the magnitude of the accidental errors. In this case *p.m.* = *d.m.* The symbol δ will be used to represent the value of *p.m.* or *d.m.* indiscriminately, as the latter is only a special case of the former when Σr^2 is negligible.

It will be shown on page 31 that any single residual r_k may be regarded as negligible in computing *p.m.* if

$$r_k \leqq \tfrac{1}{3}\, p.m. \tag{5}$$

Also that any number p of residuals are simultaneously negligible if

$$\sqrt{r_1^2 + r_2^2 + \ldots + r_p^2} \leqq \tfrac{1}{3}\, p.m. \tag{6}$$

The Probable Error and the Mean Error.—In the discussion of observations by the method of Least Squares and in many foreign treatises certain other measures are in common use; namely, the so-called *probable error* and the

mean error. The "probable error" of an observation is of such a magnitude that the probability of making an error greater than it is just equal to the probability of making one less than it, both probabilities being one-half. The probable error of a single observation and of the mean of n observations are given by the expressions

$$p.e.=0.6745\sqrt{\frac{\Sigma d^2}{n-1}} \text{ and } P.E.=0.6745\sqrt{\frac{\Sigma d^2}{n\,(n-1)}}$$

respectively, where Σd^2 is the sum of the squares of the deviations of the single observations from the mean. The following approximate formulæ are more convenient forms to use for purposes of computation:

$$p.e.=0.8453\frac{\Sigma d}{\sqrt{n\,(n-1)}} \text{ and } P.E.=0.8453\frac{\Sigma d}{n\sqrt{n-1}}\,.$$

The "mean error" μ is defined as the square root of the arithmetical mean of the squares of the errors. It is seldom used except in treatises on Least Squares.

It can be shown from the equation of the curve of error (p. 13) that, interpreted geometrically, $p.e.=OP$, the abscissa of the ordinate which divides the area OXY into equal parts; $a.d.=OD$, the abscissa of the ordinate passing through the center of gravity of the half area; and $\mu=OM$, the abscissa of the point of inflection of the curve. From this it follows that

$$p.e.=\frac{0.4769}{h}; \quad a.d.=\frac{1}{h\sqrt{\pi}}; \quad \mu=\frac{1}{h\sqrt{2}},$$

or

$$p.e.=0.85\,a.d.=0.67\mu.$$

Although the probable error is frequently used by physicists as a precision measure, the average deviation is simpler, and will be adopted throughout the present work.

Weights.—It frequently happens that it is necessary to average a series of results which have not been taken under like conditions, and which are not all equally probable; *i.e.*,

which do not have the same precision measures. In this case it is first necessary to assign relative *weights* to the observations, so that, in taking the average, the more precise measurements may be given a proportionally greater "weight" than the less precise measurements.

It can be shown that the relative weights of a series of observations are inversely proportional to the squares of their precision measures; *i.e.*, if p_1, p_2, p_3, . . . etc., are the weights of a series of observations whose respective precision measures are δ_1, δ_2, δ_3, . . . etc., respectively,

$$p_1 : p_2 : p_3 : \ldots = \frac{1}{\delta_1^2} : \frac{1}{\delta_2^2} : \frac{1}{\delta_3^2} : \ldots . \tag{7}$$

In determining the values of p, the nearest round numbers satisfying the above proportion should be chosen.

Since the various precision and deviation measures differ from each other only by a constant factor, any one of them may be used in computing "weights." It is, of course, necessary, however, that the same measure be used throughout any given discussion; *i.e.*, it is not permissible to express the precision of one quantity as an average deviation, another as a probable error, and a third as a percentage error.

The Weighted Mean.—Having obtained the weights p_1, p_2, p_3, etc., to be assigned respectively to a series of quantities m_1, m_2, m_3, etc., the best representative value or *weighted mean* will evidently be given by the expression

$$M = \frac{p_1 \times m_1 + p_2 \times m_2 + p_3 \times m_3 \ldots}{p_1 + p_2 + p_3 + \ldots} . \tag{8}$$

Rejection of Observations.—In a series of measurements taken under similar conditions, it not unfrequently happens that an observation will differ quite widely from others in the series, and the tendency to regard such an observation as erroneous and to reject it is great, particularly among beginners. If such an observation obviously contains a *mistake,* as, for example, the recording of a wrong number, the recording the wrong scale division, the incorrect adding

up of weights, etc., it may, of course, be legitimately rejected. If, however, no mistake is apparent, the observation should never be rejected without the most scrupulously unbiassed judgment on the part of the observer or the application of some mathematical criterion for the rejection of doubtful observations. For the experienced observer the former procedure is preferable, even though several mathematical criteria, Peirce's, Chauvenet's, etc., have been deduced, which are very satisfactory when the number of observations considered is large. In most physical work the number of observations is not very great, however, and one widely discordant from the others has an undue weight on the value of the mean. It is frequently better to reject such an observation, even though it contains no apparent mistake. A good criterion to follow in such cases is the following:—

Compute the mean and the average deviation *a.d.*, omitting the doubtful observation. Compute also the deviation, *d*, of the doubtful observation from the mean. If $d \geqq 4\ a.d.$, reject the observation, since it can be shown that the probability of the occurrence of an observation whose deviation is equal to four times the average deviation is only one in a thousand. An error of this unusual magnitude is called a *Huge* Error.

Computation Rules and Significant Figures.—It is probably true that at least half the time usually spent on computations is wasted, owing to the retention of more figures than the precision of the data warrants, and to the failure to use either logarithms or a slide rule instead of the lengthy arithmetical processes of multiplication and division. An important feature of physical laboratory work is the proper use of significant figures in recording data and in subsequent computations. The habit should be acquired at the outset of rejecting *at each stage of the work* all figures which have no influence on the final result.

Rules for the correct use of significant figures are discussed in the introduction to Holman's "Computation

Rules and Logarithm Tables" which may very advantageously be used in connection with the laboratory work. A fuller discussion, including the demonstration of these rules, is given in Holman's "Precision of Measurements." The following brief statement of the rules is essentially that given in these works:—

A *Digit* is any one of the ten characters 1, 2, 3, 4, 5, 6, 7, 8, 9, 0.

A *Significant Figure* is any digit to denote or signify the amount of the quantity in the place in which it stands. Thus zero may be a significant figure when it is written, not merely to locate the decimal point, but to indicate that the quantity in the place in which it stands is known to be nearer to zero than to any other digit.

For example, if a distance has been measured to the nearest fiftieth of an inch, and found to be 205.46 inches, all five of the figures, including the zero, are significant. Similarly, if the measurement had shown the distance to be nearer to 205.40 than to 205.41 or to 205.39, the *final zero* would be also significant, and *should invariably be retained,* since its presence serves the most useful purpose of showing that this place of figures had been measured as well as the rest. If in such a case the quantity had been written 205.4 instead of 205.40, the inference would be drawn either that the hundredths of an inch had not been measured or that the person who wrote the number was ignorant or careless of the proper numerical usage. Failure to follow this simple rule is a common source of annoyance and uncertainty.

A zero, when used merely to locate the decimal point, is not a significant figure in the above sense; for the position of the decimal point in any measurement is determined solely by the unit in which the quantity in question is expressed. The number of decimal places in a result has, therefore, in itself no significance in indicating the precision of a measurement. For example, suppose a certain distance is found to be 122.48 cm. with a *A.D.* of 0.12 cm. The percentage pre-

cision of the measurement is $\frac{0.12}{120.} \times 100 = 0.10\%$. The result contains five significant figures, and its precision remains the same, namely, 0.10%, whether it be expressed as 1.2248 meters, *A.D.* = 0.0012 m., or 1224.8 mm., *A.D.* = 1.2 mm. The statement that the distance is measured to 0.12 cm. gives no idea of the precision of the measurement unless the distance itself is stated. A fractional or percentage precision measure, on the other hand, gives a definite idea of the precision of the measurement without any further statement, as it involves both the value of the quantity and its average deviation.

The following rules are deduced subject to the condition that the accumulated errors in a computation shall not affect the second unreliable place of figures in the final result by more than one unit, even though as many as sixteen rejections of figures are made in the course of the computation. This is a safe limit to assume for most physical work, as it is seldom that more than this number of quantities or operations enter into any single computation.

Rule I.—In rejecting superfluous figures, increase by 1 *the last figure retained, if the following figure (that rejected) is* 5 *or over.*

Rule II.—In all deviation and precision measures retain two, and only two, significant figures.

The reason for this rule is as follows: consider the above example where the length measured is $m = 122.48$ cm. with an *A.D.* = 0.12 cm. The significance of the *A.D.* is that the place of figures in m occupied by the 4 is uncertain by 1 unit, and that the next place of figures occupied by 8 is uncertain by 12 units, while the third decimal place would be uncertain by at least 120 units; *i.e.*, by an amount which would render it practically worthless. In general, the place of figures corresponding to the *first* significant figure of the deviation measure is somewhat uncertain (from 1 to 9 units), while the place corresponding to the *second* significant figure in the deviation measure is uncertain by ten times this amount (10 to 90 units, or, more exactly, 10 to 99 units). Beyond

this place the significance of additional figures is so slight as to be of no value: hence, as deviations and precision measures are at best only estimates of the reliability of a result, it is useless to compute them to places of figures which have no real significance in the result to which they refer.

If the first significant figure of the precision measure is as great as 8 or 9, in which case the place of figures in the data corresponding to the second place in the precision measure is unreliable by 80 to 90 units, it is usually sufficient to retain but one significant figure in the precision measure.

Rule III.—Retain as many places of figures in a mean result and in data in general as correspond to the second place of significant figures in the deviation or precision measure.

Two places of doubtful figures are thus retained in data and computations rather than one, so that accumulated errors due to rejections in the course of a computation may not affect the first place of uncertain figures in the result.

Rule IV.—The sum or difference of two or more quantities cannot be more precise numerically than the quantity having the largest average deviation. Hence, *in adding or subtracting a number of quantities, find the average deviation of each, and then retain in each quantity as many places of figures as correspond to the second place of significant figures in the largest deviation.*

Rule V.—In multiplication or division the *percentage* precision of the product or quotien cannot be greater than the *percentage* precision of the least precise factor entering into the computation. Hence, in *computations involving these operations, the number of significant figures to be retained in each factor is determined by the number properly retained under Rule III. in the factor which has the largest percentage deviation.* Computations involving a precision not greater than ¼ per cent. should be made with a slide rule. For greater precision logarithm tables should be used. If multiplication and division must be resorted to, the "short method" of rejecting all superfluous figures at each stage of the operation should be adopted.

Rule VI.—In carrying out the operations of multiplication and division by logarithms, retain as many figures in the mantissa of the logarithm of each factor as are properly retained in the factors themselves under Rule V.

Precision Discussion of Indirect Measurements.—We will now consider the precision discussion of indirect measurements; *i.e.*, those in which the final result is a more or less complicated function of one or more directly measured quantities. Two distinct classes of problems may arise:

First.—The precision measures of the directly measured components are known (determined as above described), and it is desired to find the precision measure of the final result.

Second.—The desired precision of the final result is stipulated at the outset, and the problem is to ascertain what precision is necessary in the components, in order that the accumulated effect of the errors in these on the final result shall not exceed the prescribed limit.

The importance of these problems cannot be overestimated; for, in the first case, a final result, be it the result of chemical analysis, the value of a physical constant, the algebraic expression of a law, or an efficiency test of an engine, is practically worthless unless a numerical estimate of its reliability can be stated. In fact, it may be worse than worthless if carried out to indicate a higher precision than the data warrant. And the second case is of equal importance; for, unless an investigator makes a preliminary precision discussion of his method and apparatus before beginning work, so that he may know, at least approximately, how precisely each quantity entering into the final result should be measured, the chances are that much time and labor will be wasted in measuring some components more precisely than necessary, while others will be measured to a degree of precision which will render impossible the attainment of the desired precision in the final result.

Notation.—In the precision discussion which follows, the following notation will be adopted.

M = the final computed result of any indirectly measured quantity.

Δ = numerical precision measure of M.

$m_1, m_2, \ldots$ = directly measured quantities, which may be either mean results or single observations.

$\delta_1, \delta_2, \ldots$ = the numerical precision measures of $m_1, m_2, \ldots$ respectively.

The values of δ might be expressed as average deviations, probable errors, or mean errors, discussed on page 19. In the discussion of any given problem, however, the same kind of precision measure must be used throughout; *i.e.*, in any given problem it is not proper to express the precision measures of some quantities as probable errors, others as average deviations, and still others as percentage or fractional deviations. *In the following discussion we shall always assume values of δ to be expressed as deviations.*

$\Delta_1, \Delta_2, \ldots$, will be used to denote the deviations produced in M by deviations $\delta_1, \delta_2, \ldots$ in the components $m_1, m_2, \ldots$ respectively.

From the above notation it follows that,

$\frac{\Delta}{M}$ = the fractional precision of the final result;

$100\frac{\Delta}{M}$ = percentage precision of the final result;

$\frac{\delta_1}{m_1}, \frac{\delta_2}{m_2}, \ldots$ = the fractional precision of the components $m_1, m_2, \ldots$ respectively;

$100\frac{\delta_1}{m_1}, 100\frac{\delta_2}{m_2}, \ldots$ = the percentage precision of the components $m_1, m_2, \ldots$ respectively.

In general

$$M = f(m_1, m_2, \ldots m_n), \tag{9}$$

which for brevity may be written $M = f(\ \ )$, where the form of the function is determined by the formula by which M is computed from m_1, m_2, etc. The first class of problems may then be stated mathematically as follows:—

Case I.—The Direct Problem. *Given the precision measures* $\delta_1, \delta_2, \ldots \delta_n$, *of the component measurements* $m_1, m_2, \ldots m_n$, *to compute the precision measure* Δ *of the result* M.

The solution of this problem is obtained by finding, first, the effect of the deviation in each component on M, and then combining these separate effects to get the resultant effect. The method of computation to be followed in this last procedure depends upon the law to which the deviations concerned are subject, and will be considered below.

Separate Effects.—The effect of a deviation δ_k in any component m_k will be to produce a deviation Δ_k in M of an amount

$$\Delta_k = \frac{\partial M}{\partial m_k} \cdot \delta_k,$$

$$= \frac{\partial}{\partial m_k} f(\quad) \cdot \delta_k, \qquad (10)$$

i.e., an amount equal to the rate at which the value of the function $M = f(\quad)$ changes, as m_k changes (the other components $m_2, m_3, \ldots$ etc., remaining constant), multiplied by the actual change δ_k in m_k, or, in other words, the partial differential coefficient of the function with respect to m_k multiplied by the actual deviation δ_k in m_k.

Example 1.—Find the deviation in the volume of a sphere whose diameter is 10.013 cm., if the average deviation in the measurement of the diameter is $A.D. = 0.012$ cm.

$$V = \tfrac{1}{6}\pi D^3.$$

Comparing with the notation on page 26, it is evident that

$$M = V = f(\quad) = \tfrac{1}{6}\pi D^3,$$

$$m = D = 10.013 \text{ cm., and } \delta = A.D. = 0.012 \text{ cm.}$$

The computed value of V is

$$V = \tfrac{1}{6} \times 3.1416 \times \overline{10.013}^3$$

$$= 525.52 \text{ c.c.}$$

By (10) the deviation Δ in this volume produced by the deviation δ in the diameter is

$$\Delta = \frac{d}{dD}(\tfrac{1}{6}\pi D^3) \cdot \delta$$

$$= \tfrac{1}{6}\pi \cdot 3D^2 \cdot \delta$$

$$= \tfrac{1}{6} \times 3.1 \times 3 \times \overline{10}^2 \cdot 0.012$$

$$= 1.9 \text{ c.c.};$$

i.e., the volume 525.5 c.c. is uncertain by 1.9 c.c., or by 19 parts in 5300. A deviation of $100\,\frac{\delta}{D} = 100\,\frac{0.012}{10} = 0.12\%$ in the diameter introduces a deviation of $100\,\frac{\Delta}{V} = 100\,\frac{1.9}{530} = 0.36\%$ in the volume, *i.e.*, a percentage deviation three times as great. In this case V is a function of only a single variable, hence the resultant deviation in V is given at once by the above result.

Example 2.—What will be the numerical deviation in the value of g, as determined by a second's pendulum, due to a deviation $A.D. = 0.0020$ second in the determination of the time of vibration, and a deviation $A.D. = 0.10$ cm. in the determination of the length?

$$g = \frac{\pi^2 l}{t^2}.$$

Hence in the general notation $M = g = f\,(\quad) = \frac{\pi^2 l}{t^2}$; $m_1 = l = 100$ cm.; $m_2 = t = 1$ sec.; $\delta_1 = \delta_l = 0.10$ cm.; $\delta_2 = \delta_t = 0.0020$ sec.

The deviation Δ_l in g produced by the deviation δ_l in l is by (10)

$$\begin{aligned}\Delta_l &= \frac{\partial}{\partial l}\left(\frac{\pi^2 l}{t^2}\right) \cdot \delta_l \\ &= \frac{\pi^2}{t^2} \times 1 \times \delta_l \\ &= \frac{\overline{3.1}^2}{1^2} \times 1 \times 0.10 \\ &= 0.96\,\frac{\text{cm.}}{\text{sec.}^2};\end{aligned}$$

i.e., a deviation of 0.10 cm. in the measurement of l will produce a deviation of $0.96\,\frac{\text{cm.}}{\text{sec.}^2}$ in the value of g.

Similarly the deviation Δ_t in g due to the deviation δ_t in t is by (10)

$$\begin{aligned}\Delta_t &= \frac{\partial}{\partial t}\left(\frac{\pi^2 l}{t^2}\right) \cdot \delta_t \\ &= -2\,\frac{\pi^2 l}{t^3} \cdot \delta_t \\ &= -\frac{2 \times \overline{3.1}^2 \times 100}{1^3} \times 0.0020 \\ &= -3.8\,\frac{\text{cm.}}{\text{sec.}^2};\end{aligned}$$

i.e., a deviation of 0.0020 sec. in the measurement of the time will introduce an uncertainty in the value of g of $3.8 \frac{\text{cm.}}{\text{sec.}^2}$. The negative sign simply indicates that a positive deviation in t produces a negative deviation in g, and *vice versa*. Since all deviations are equally likely to be plus or minus, in precision discussions no attention is usually paid to the sign resulting from differentiation of a function. By a direct application of (10) the effect of a deviation in any single component on a final result may always be computed. A much shorter method than the above, applicable in certain special cases, will be pointed out below.

Resultant Effect.—To find the combined or resultant effect Δ, *on the final result of the separate deviations* Δ_1, Δ_2, . . . etc., *produced by the components.*

If for any reason the values of Δ_1, Δ_2, . . . etc., are of specified magnitude and sign (in which case they would not follow the general law of deviations), they should be combined according to the formula

$$\Delta = \Delta_1 + \Delta_2 + \ldots + \Delta_n. \tag{11}$$

As this case rarely occurs in practice, it need not be further discussed here.

The important case to consider is that in which the values of Δ_1, Δ_2, . . . etc., are equally likely to be plus or minus and of a magnitude determined by the general law of deviations. If each Δ_k is computed by formula (10), page 27, *i.e.*,

$$\Delta_k = \frac{\partial}{\partial m_k} f(\quad) \cdot \delta_k,$$

these conditions will always be fulfilled, since the values of δ_k which determine Δ_k are of the nature of true deviations. Under these circumstances the *most probable* resulting deviation Δ, in M, can be shown by the method of Least Squares to be that obtained by combining the values of Δ_k by the formula

$$\Delta^2 = \Delta_1^2 + \Delta_2^2 + \ldots + \Delta^2_n, \tag{12}$$

$$\text{or} \quad \Delta = \sqrt{\Delta_1^2 + \Delta_2^2 + \ldots + \Delta_n^2}. \tag{12a}$$

This does not give us an exact solution of the problem, but rather the solution which in the long run is better than that obtained by any other method of combining the values of Δ_k. It is to be noted that by this method of computation the effect of the sign of individual deviations Δ_k is eliminated. The resultant deviation Δ is, of course, to be regarded as equally likely plus or minus.

Example 2 (continued).—Thus in Example 2 the combined effect of the deviations in l and in t on the value of g is to be found by taking the square root of the sum of the squares of the deviations Δ_l and Δ_t, which δl and δt separately produce in g respectively; *i.e.*,

$$\begin{aligned}\Delta &= \sqrt{\Delta_l^2 + \Delta_t^2}\\ &= \sqrt{0.96^2 + 3.8^2}\\ &= 3.9\,\frac{\text{cm.}}{\text{sec.}^2}\end{aligned}$$

Hence a deviation of 0.10 cm. in l and 0.0020 sec. in t will make the value of $g = 980\,\frac{\text{cm.}}{\text{sec.}^2}$ uncertain by nearly $4\,\frac{\text{cm.}}{\text{sec.}^2}$.

Criterion for Negligibility of Deviations in Components.—It is frequently important to determine whether the deviation arising from one or more components may be neglected in computing the Δ of the final result. For this purpose the following criterion may be deduced.

As explained under rules for significant figures on page 23, two significant figures are all that should be retained in any deviation measure. A quantity which affects a result by only $\frac{1}{10}$ the amount of its deviation or precision measure will therefore affect it only in that place of significant figures corresponding to the second place in the deviation measure. This place is so uncertain that such an amount may in general be regarded as negligible. Although the assumption that $\frac{1}{10}$ *p.m.* or $\frac{1}{10}$ *d.m.* is negligible is somewhat arbitrary, it has been found to be a convenient and practical criterion to adopt.

Suppose, therefore, that the value of the Δ of some quantity

M is made up of deviations $\Delta_1, \Delta_2, \ldots \Delta_n$, arising from various deviations $\delta_1, \delta_2, \ldots \delta_n$, in components $m_1, m_2, \ldots m_n$. May any of these Δ's, as Δ_k, be neglected in computing Δ, or, in other words, may any of the components, as m_k, be regarded as being without sensible error on M?

To answer this question, let

$$\Delta = \sqrt{\Delta_1^2 + \Delta_2^2 + \ldots + \Delta_k^2 + \ldots \Delta_n^2}$$

$$\text{and } \Delta' = \sqrt{\Delta_1^2 + \Delta_2^2 + \ldots + \Delta_n^2} \text{ with } \Delta_k \text{ omitted.}$$

Then, if $\quad \Delta - \Delta' \overline{\overline{<}} \tfrac{1}{10} \Delta$

$$\text{or } \Delta' \overline{\overline{>}} 0.9\,\Delta,$$

by the above criterion Δ_k may be considered as negligible.

$$\begin{aligned} \text{But } \Delta_k^2 &= \Delta^2 - \Delta'^2 \\ &= \Delta^2\,(1^2 - 0.9^2) \\ &= 0.19\,\Delta^2 \\ \therefore\ \Delta_k &= 0.43\,\Delta. \end{aligned}$$

Hence the deviation in any component m may be neglected in computing the Δ of M, if its effect on M is equal or less than $0.43\,\Delta$. A still safer and more convenient criterion to adopt, since the number of components considered is usually small and hence the assumed formula of squares is less rigidly applicable, is

$$\Delta_k \overline{\overline{<}} 0.33\,\Delta \overline{\overline{<}} \tfrac{1}{3}\,\Delta. \tag{13}$$

In the same way it can be shown that deviations in any number, p, components are simultaneously negligible if

$$\sqrt{\Delta_1^2 + \Delta_2^2 + \ldots \Delta_p^2} \overline{\overline{<}} \tfrac{1}{3}\,\Delta. \tag{14}$$

The above criterion also applies to the rejection of residuals in computing the value of the precision measure by the formula $\overline{p.m.} = \sqrt{\overline{d.m.}^2 + r_1^2 + r_2^2 + \ldots r_n^2}$, as stated on page 18.

Case II.—The Converse Problem. *Given a prescribed precision Δ to be attained in the final result M, to find the allowable deviations δ_1, δ_2, etc., in the components m_1, m_2, etc., re-*

spectively, such that their combined effect on M shall not exceed the value of Δ.

We have seen that when the deviations follow the law of errors,

$$\Delta^2 = \Delta_1^2 + \Delta_2^2 + \ldots + \Delta_n^2,$$

$$\text{where } \Delta_k = \frac{\partial}{\partial m_k} f(\quad) \cdot \delta_k$$

$$\text{or } \delta_k = \Delta_k \div \frac{\partial}{\partial m_k} f(\quad). \tag{15}$$

If the value of Δ is given and no further conditions imposed, there are evidently an infinite number of solutions to the problem; *i.e.*, an indefinite number of values can be found for Δ_1, Δ_2, . . . etc. (and hence for the corresponding values of δ_1, δ_2, . . . etc.), which will satisfy the above equation.

The most advantageous distribution of errors among the components will evidently be that one by which the desired precision is obtained with the minimum expenditure of time and labor on the part of the experimenter. As this will vary greatly with each individual problem, no mathematical criterion can be formulated which will embrace all cases. It is best, therefore, at least for a preliminary distribution of errors among the components, to so adjust them that the errors inherent in each variable or component shall produce the *same* effect on the final result. This is spoken of as the solution of the problem for "*equal effects.*"

Solving the formula for resultant effects subject to this condition, *i.e.*,

$$\Delta_1^2 = \Delta_2^2 = \ldots = \Delta_k^2 = \ldots = \Delta_n^2,$$

we have

$$\Delta^2 = n\,\Delta_k^2,$$

hence for any component,

$$\Delta_k = \frac{\Delta}{\sqrt{n}}. \tag{16}$$

Having thus determined Δ_k, the corresponding value of δ_k can be found at once by equation (15).

Example 3.—How precisely should the time of vibration and length of a seconds pendulum be measured in order that the computed value of g may be reliable to one-tenth of one per cent.? $g = \frac{\pi^2 l}{t^2}$.

As the pendulum is stated to be a "seconds" pendulum, t = one second and $l = 100$ cm. approximately. It is further stipulated that $100 \frac{\Delta}{g} \overline{\gtrless} 0.10$; hence the allowable resultant deviation Δ in g must not be greater than $\Delta = g \times 0.0010 = 0.98 \frac{\text{cm.}}{\overline{\text{sec.}}^2}$. We are to find the allowable values of δ_t and δ_l, which will give this precision. Solving the problem subject to the condition of equal effects,—*i.e.*, that the resultant deviation in g is caused equally by the deviation in t and in l,—we have

$$\Delta_t = \Delta_l = \frac{\Delta}{\sqrt{n}} = \frac{0.98}{\sqrt{2}} \frac{\text{cm.}}{\overline{\text{sec.}}^2} = 0.70 \frac{\text{cm.}}{\overline{\text{sec.}}^2}.$$

Hence the allowable deviations in the time and length measurements must be reduced to such a magnitude that they do not separately produce a deviation in g greater than $0.70 \frac{\text{cm.}}{\overline{\text{sec.}}^2}$ respectively. But by the general equation (10), page 27

$$\Delta_t = \frac{\partial g}{\partial t} \cdot \delta_t = \frac{\partial}{\partial t}\left(\frac{\pi^2 l}{t^2}\right) \cdot \delta_t = \frac{2\pi^2 l}{t^3} \cdot \delta_t$$

or

$$0.70 \frac{\text{cm.}}{\overline{\text{sec.}}^2} = \frac{2 \times \overline{3.1}^2 \times 100 \text{ cm.}}{1^3 \, \overline{\text{sec.}}^3} \cdot \delta_t$$

hence $\delta_t = 0.00037$ sec.

Similarly,

$$\Delta_l = \frac{\partial g}{\partial l} \cdot \delta_l = \frac{\partial}{\partial l}\left(\frac{\pi^2 l}{t^2}\right) \cdot \delta_l = \frac{\pi^2}{t^2} \cdot \delta_l$$

or

$$0.70 \frac{\text{cm.}}{\overline{\text{sec.}}^2} = \frac{\overline{3.1}^2}{1^2 \, \overline{\text{sec.}}^2} \cdot \delta_l$$

hence $\delta_l = 0.073$ cm.

The time of vibration of the pendulum should therefore be measured to 0.00037 sec., and its length measured to 0.073 cm.

The Fractional or Percentage Method of Solution.—The preceding formulæ for obtaining the precision of a final result from the known or estimated precision of the com-

ponent measurements, and for calculating the necessary precision of the component measurements when the desired precision of the final result is stipulated, are entirely general, and by them any type of problem can be solved. It is to be particularly noted throughout the preceding discussion that the values of the precision or deviation measures δ and Δ are numerical deviations expressed in the same units as the quantities to which they refer. Percentage and fractional deviations should not be used when applying the general formulæ (10) to (12*a*), (15) and (16). If in the statement of a problem, as in example 3, the fractional or percentage precision is given, the corresponding deviations δ or Δ should first be computed before proceeding with the solution.

There are, however, a large number of formulæ which may be discussed with a great saving of time and labor by the use of percentage or fractional deviations. This is the case whenever the function $M = f(m_1, m_2, \ldots m_n)$ can be put in the form of a product of the general type

$$M = k \cdot m_1^a \cdot m_2^b \ldots m_n^p \tag{17}$$

where k, a, b, $\ldots p$ are constants (positive, negative, fractional, or integral). For all such cases a very simple relation holds between the fractional or percentage deviation in any component and the fractional or percentage deviation which it produces in the final result. This may be shown as follows. Applying the general formula (10) for separate effects to the above special case, we have for the deviation Δ_1 in M produced by δ_1 in m_1

$$\Delta_1 = \frac{\partial M}{\partial m_1} \cdot \delta_1 = (k \cdot m_2^b \ldots m_n^p) \cdot a m_1^{a-1} \delta_1.$$

Dividing through by equation (17)

$$\frac{\Delta_1}{M} = a \cdot \frac{\delta_1}{m_1} \tag{18}$$

i.e., a fractional deviation $\frac{\delta_1}{m_1}$ in m_1 produces a fractional

deviation a times as great in the final result. Thus, if $a=2$, a deviation of one per cent. $\left(100\,\frac{\delta_1}{m_1}=1\right)$ in m_1 will introduce a deviation of two per cent. in M, no matter what the value of the remaining factors in the expression may be. The separate effect of a known fractional or percentage deviation in any component on the final result may therefore be stated at once by inspection, whenever the formula under discussion can be put in the above form.

Since the formula for Resultant Effects (12a) may be put in the form

$$\frac{\Delta}{M}=\sqrt{\left(\frac{\Delta_1}{M}\right)^2+\left(\frac{\Delta_2}{M}\right)^2+\ldots\left(\frac{\Delta_n}{M}\right)^2} \tag{19}$$

the complete solution for any product function of the type given by equation (17) may be written down at once by inspection as

$$\frac{\Delta}{M}=\sqrt{\left(a\frac{\delta_1}{m_1}\right)^2+\left(b\frac{\delta_2}{m_2}\right)^2+\ldots\left(p\frac{\delta_n}{m_n}\right)^2} \tag{20}$$

if the fractional deviations $\frac{\delta_1}{m_1}$, etc., of the components are known.

Similarly, the solution of the converse problem for a product function is equally simple, as the condition for equal effects, page 32, may be written

$$\frac{\Delta_1}{M}=\frac{\Delta_2}{M}=\ldots=\frac{\Delta_k}{M}=\ldots=\frac{\Delta_n}{M} \tag{21}$$

and hence the allowable fractional deviation in the final result which any component as m_k may produce is

$$\frac{\Delta_k}{M}\overline{\overline{<}}\frac{1}{\sqrt{n}}\cdot\frac{\Delta}{M} \tag{22}$$

where $\frac{\Delta}{M}$ is the prescribed fractional deviation of the

final result which must not be exceeded. Having thus determined the value of $\frac{\Delta_k}{M}$, we obtain at once the allowable fractional deviation $\frac{\delta_k}{m_k}$ in the corresponding component m_k by inspection from the simple relation expressed by equation (18).

Since the exponents a, b, c, etc., of the factors may have negative as well as positive values, the above solutions apply to formulæ involving division as well as multiplication of factors.

It is to be especially noted that, if the function M involves the sum or difference of several components, or is a trigonometric or logarithmic function, no simple relation exists between the fractional deviation of a component and the fractional deviation which it produces in the final result. The above special method of procedure is, therefore, inapplicable to such cases. This will be readily seen from the following simple example. Suppose

$$M = a\ m_1 + b\ m_2.$$

Then
$$\Delta_1 = \frac{\partial M}{\partial m_1} \cdot \delta_1 = a \cdot \delta_1$$

and
$$\frac{\Delta_1}{M} = \frac{a\delta_1}{a\ m_1 + b\ m_2},$$

from which it appears that $\frac{\Delta_1}{M}$ stands in no simple relation to $\frac{\delta_1}{m_1}$ unless the term $b\ m_2$ happens to be negligible in magnitude compared with $a\ m_1$, in which case we should have assumed at the outset for our precision discussion that $M = a\ m_1$ approximately.

It frequently happens, however, that apparently complicated functions can be transformed into a simple product of factors by changing variables or noting that certain components may be neglected in the precision discussion. When this is possible, the fractional method may be applied with advantage to each factor.

Example 4.—The solution of problem 2 may be obtained much simpler by the fractional or percentage method than by the general method as worked out on page 28. For the formula $g = \pi^2 \frac{l}{t^2} = \pi^2 l\, t^{-2}$ is evidently a simple product function of the variables l and t. To find the deviation in g due to a deviation of $\delta_l = 0.10$ cm. in l and a deviation $\delta_t = 0.0020$ second in t, we find first the *fractional* deviation in l and in t respectively.

$$\frac{\delta_l}{l} = \frac{0.10 \text{ cm.}}{100 \text{ cm.}} = 0.0010,$$

and
$$\frac{\delta_t}{t} = \frac{0.0020 \text{ sec.}}{1.0 \text{ sec.}} = 0.0020.$$

Then by inspection, since g is directly proportional to the first power of l,

$$\frac{\Delta_l}{g} = \frac{\delta_l}{l} = 0.0010;$$

and, since g is proportional to the second power of t (neglecting sign),

$$\frac{\Delta_t}{g} = 2\frac{\delta_t}{t} = 2 \times 0.0020 = 0.0040.$$

Hence
$$\frac{\Delta}{g} = \sqrt{\left(\frac{\Delta_l}{g}\right)^2 + \left(\frac{\Delta_t}{g}\right)^2}$$
$$= \sqrt{(0.0010)^2 + (0.0040)^2} = \pm 0.0041$$

or
$$\Delta = \pm 0.0041 \times 980 \frac{\text{cm.}}{\text{sec.}^2} = \pm 4.0 \frac{\text{cm.}}{\text{sec.}^2}$$

which is practically the same result previously obtained, the slight difference arising from the use of but two significant figures in the computation.

Example 5.—Again, the solution of example 3, page 33, may be simplified by using the fractional method. Thus, if it is stipulated that g is to be measured to 0.10 per cent., *i.e.*, $100 \frac{\Delta}{g} \leqq 0.10$, the prescribed fractional deviation is $\frac{\Delta}{g} \leqq 0.0010$. Distributing this deviation by the criterion of equal effects between the component measurements l and t respectively, we have

$$\frac{\Delta_l}{g} = \frac{\Delta_t}{g} = \frac{1}{\sqrt{2}} \cdot \frac{\Delta}{g} = \frac{0.0010}{\sqrt{2}} = 0.00071.$$

But by inspection of the formula $g = \frac{\pi^2 l}{t^2}$ it is seen that g is

proportional to the first power of l and to the second power of t, therefore

$$\frac{\Delta_l}{g} = \frac{\delta_l}{l} \text{ and } \frac{\Delta_t}{g} = 2\frac{\delta_t}{t}.$$

Hence

$$\frac{\delta_l}{l} = 0.00071,$$

or the length must be measured to

$$\delta_l = 100 \text{ cm.} \times 0.00071 = 0.071 \text{ cm.}$$

Similarly,

$$\frac{\delta_t}{t} = \frac{1}{2} \times 0.00071 = 0.00036,$$

or the time must be measured to

$$\delta_t = 1 \text{ sec.} \times 0.00036 = 0.00036 \text{ sec.}$$

These, it is seen, are practically the same values previously obtained by the differentiation method on page 33, the difference in the second place of figures arising from the use of only two figures in the computation.

Discussion of "Equal Effect" Solution.—It not infrequently happens upon solving a problem as described above, that some component (or components) can with little additional time and labor be determined with a much higher precision than the solution demands. In this case such a component or factor should be so measured and then regarded as a constant in the precision discussion, since the error in it will have a negligible effect on the final result. The problem should then be re-solved on the basis of one less variable, in which case the remaining components which are more difficult to determine, may be measured with somewhat less precision than was demanded by the first solution.

The proper adjustment of precision among components so as to give the desired precision in the final result with the apparatus at one's disposal and with the least expenditure of time and labor, requires some experience and good judgment on the part of the investigator. Beginners will not go far astray, however, if they follow the above criterion for equal effects.

PART II.

GRAPHICAL METHODS.

GRAPHICAL METHODS.

Nature of Problems.—The graphical method of discussing experimental data is of great convenience and importance when the problem under investigation is to determine the law or fundamental relationship between two quantities. This type of problem arises very frequently in scientific and technical investigations. The graphical method is also of great value for purposes of interpolation, discussion of corrections, etc.

Procedure.—The general procedure to be followed in discussing observations by the graphical method will be explained and illustrated by following through, step by step, a specific problem. Suppose it is desired to find the relation which holds between the resistance of a certain coil of wire and its temperature, between 10° and 100° C.; that is, to determine the formula by which the resistance can be computed at any given temperature between these limits. The experimental procedure would consist in making a series of measurements of the resistance r of the wire at various temperatures t from approximately 10° to 100° C. Suppose that the result of such experiments gives the two following columns of data, the resistance measurements being reliable to 0.003 ohm, and the temperature measurements to 0.02° C., as shown by their respective deviation or precision measures.

EXPERIMENTAL DATA.

r = *resistance of coil in ohms.*	t = *temperature of coil in degrees C.*
10.421	10.50
10.939	29.49
11.321	42.70
11.799	60.01
12.242	75.51
12.668	91.05

The Direct Plot.—To obtain some clue to the relation between r and t (supposing it unknown), a *Direct Plot* should first be made. Plotting-paper suitable for this work should be ruled with carefully adjusted pens, otherwise the errors arising from irregularity of ruling may easily exceed those of only moderately accurate data. A convenient size is about eight by ten inches, ruled either in millimeters, or preferably, in twentieths of an inch.

First.—Choice of Ordinates and Abscissæ. The first thing to decide upon is which data are to be plotted as ordinates and which as abscissæ. The usual convention of analytic geometry should always be followed. If, as in the problem under consideration, it is desired to obtain a relation in which r is expressed as a function of t, then values of r should be plotted as ordinates and values of t as abscissæ. If, on the other hand, it were desired to obtain a formula for computing the temperature t corresponding to any resistance r, as in resistance pyrometry, the converse would be the case.

Second.—Choice of Scales. By the "scale" of a plot is meant the ratio of the number of units (inches, centimeters, etc.) of the plot to one unit of the data. Scales of both ordinates and abscissæ should be clearly indicated on the plot. Thus, if 100° is plotted so as to extend over 10 inches, the scale is $10'' : 100° = 1 : 10$, or one-tenth. This is usually expressed as 1 inch to 10 degrees. In general, it is not feasible to choose the same scale for both ordinates and abscissæ, nor should the attempt be made to have the origin fall on the plot. If equal scales are chosen for both abscissæ and ordinates, the locus of the data is likely to be a line either nearly horizontal, in which case the precision of the data plotted as ordinates is sacrificed, or nearly vertical, in which case the same is true of the abscissæ. Moreover, the intersection of a nearly horizontal line with lines parallel to the axis of X can be read off only with difficulty and liability to error, while its point of intersection with the axis of Y is much

more definitely defined. In order, therefore, to preserve equal precision in the interpolation of both co-ordinates, the line should be inclined as nearly as may be at an angle of 45° with both axes. Deviations of 10° or so to either side of this position are not serious.

The scales chosen should, furthermore, be convenient; *i.e.*, in aiming to distribute the data approximately 45° across the plotting-paper, scales of one inch equal to 1, 2, 4, 5, or 10 units (or these units multiplied by $10^{\pm n}$ where n is an integer), should be chosen, but never such scales as one inch to 3, 7, 6, 11 units. The latter scales make plotting not only laborious, but very liable to error, whereas the former scales permit data to be plotted with facility. In choosing scales for plotting, the student should guard as carefully against adopting excessively large scales as excessively small ones. In the latter case the plot will be cramped and the precision of the data sacrificed. In the former case the deviations of the data from the general law which they follow are likely to be so magnified to the eye that it is difficult or impossible to draw a representative line. Moreover, such plots give an exaggerated idea of the precision of the data. As an *upper limit*, a safe rule to follow is to adopt a scale which permits of easy interpolation to not more than two uncertain places of figures in the data; *i.e.*, to that place corresponding to the second significant figure in the deviation or precision measure. This rule applies particularly to data extending over narrow numerical limits, to corrections, etc.

In the present problem it is seen that the extreme variation of r is about 2.3 ohms, and of t, 90°. The scales should therefore be so chosen as to distribute these quantities well over the paper. Scales of $1'' = 0.4$ ohm and $1'' = 10°$ evidently fulfill this condition and are at the same time convenient. It is evident, however, that some of the precision of the data will be sacrificed in plotting with these scales, since it is impossible on a plot of the size chosen to locate the last significant figure of the data with any great degree of precision. It should also be noticed that the origin will not fall

on the plot. This is not at all necessary, and only in those cases when the data for both variables simultaneously approach small values (zero) is this likely to be the case. It is, however, desirable (although not imperative), that the zero value of the abscissæ should fall on the plot, in order to determine the intercept of the curve with the ordinate through this point for reasons explained below.

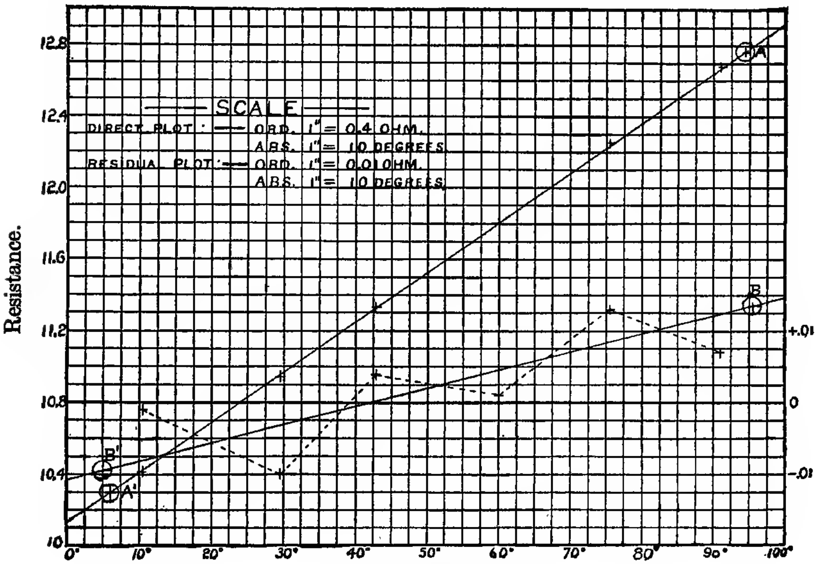
Third.—To plot the data. Data should be plotted as follows. Locate the abscissa of the first point along the axis of abscissæ and with a straight edge placed vertically through this point draw a fine line about one-eighth of an inch long approximately at the place where the corresponding ordinate is to be located. Then locate the ordinate along the axis of ordinates, and draw a short horizontal line intersecting the first line drawn. The intersection is the desired position of the point. Never locate the data by dots, as the precision attainable with the paper is not only sacrificed, but there is also much greater liability to error in the operation of plotting itself when the attempt is made to locate both ordinate and abscissa at the same time.

Fourth.—To draw the "Best Representative Line." The data being plotted, the next step is to draw a smooth curve, the equation of which shall best represent the law connecting the two variables in question. Inspection of the general form of this curve will usually give valuable information as to the form of the equation sought.

If the points appear to lie along a straight line, the *best representative line* may be located by moving a stretched fine black thread among the points until a position is found such that the points lie as nearly as may be alternately on either side of the thread and in such a manner that the points above the line deviate from it by the same amount as those below. The exact criterion for locating the best representative line would be to so adjust it among the points that the sum of the squares of the deviations of the points above the line is equal to the sum of the squares of the deviations of the points below the line. (Criterion of Least Squares.) When

the best position of the thread has been found, the location of two points through which it passes is noted, and a fine straight line is then ruled through these points with a hard pencil, or, better, with a ruling pen.

If the points cannot be uniformly distributed about a straight line, but deviate systematically from it, then the best representative curved line is to be drawn with a French or a flexible curve. The line should in this case be drawn as before, so that the points are distributed as nearly as may



PLOT I.

be on alternate sides. From the form of the resulting curve its equation may often be inferred. The next step is to determine the equation of the curve by transforming it graphically into a straight line by some one of the special methods of transformation described below. The numerical constants in the equation of a straight line can always be readily determined directly from the plot.

In the problem under consideration, the points are seen, Plot I., to lie very closely along a straight line $A'A''$, the

deviations from the line being of an irregular and not of a systematic character. The relation between r and t is therefore a linear one; *i.e.*, of the first degree. To determine completely the function $r = f(t)$, we have to find the numerical value of the constants in the equation of this line $A'A''$.

Determination of the Constants of a Straight Line.—The general equation of a straight line is

$$y = ax + b, \tag{1}$$

where a and b are constants.

The constant $a = \frac{dy}{dx}$ is the tangent of the angle θ which the line makes with the axis of X. The value of a cannot in general be found by reading off the angle with a protractor and looking out the value of its natural tangent, as the angle is usually distorted owing to the unequal scales used in plotting. To determine a, read off the value of the ordinate and abscissa x', y' and x'', y'', respectively, of any two points *on the line*, preferably near the extremities. These points will not in general be observed points. Then

$$a = \tan\,\theta = \frac{y'' - y'}{x'' - x'}.$$

Thus the co-ordinates of two such points A' and A'' are seen to be $x' = 6.0°$, $y' = 10.30$ ohms, and $x'' = 94.5°$, $y'' = 12.76$ ohms. Hence

$$a = \tan\theta = \frac{12.76 - 10.30}{94.5 \quad - \quad 6.0} = \frac{2.46}{88.5} = 0.0278.$$

The constant b is the value of y when $x = 0$; that is, it is the intercept of the line (prolonged, if necessary) on the axis of Y, read off on the scale of ordinates chosen. Thus from the plot it is seen that the line $A'A''$ cuts the ordinate through $x = 0°$ at $b = 10.13$. The desired equation connecting r and t is therefore

$$r = 0.0278\,t + 10.13. \tag{2}$$

Whenever the data are such that a long extrapolation of the line is necessary in order to make it cut the ordinate through $x = 0$, or when this ordinate falls off the plot, the value of b is found as follows. Substitute the value of a as determined above, together with values of x' and y' of some point on the line, in equation (1) and solve for b directly.

It is to be noted here that the precision of the constants in equation (2) is less than the precision of the original data. Values of r computed by this formula cannot at best be more precise than one or two parts in 1,000, while the observed values were stated to be reliable to 3 parts in 10,000; in other words, the full precision of the data has not been utilized in the plot of the size here chosen. The procedure by means of which the precision of the constants as above determined may be increased, and another place of significant figures obtained, will be explained later. See Residual Plot, p. 60.

Rectification of Curved Lines.—When the plotted data do not lie along a straight line, the form of the smooth curve best representing the points will often suggest the relationship sought. Thus curves resembling any of the conic sections or trigonometric functions are usually readily recognized. In all such cases it is usually necessary to transform the curve into a straight line in order to determine the constants in its equation. Suppose from inspection of the curve that the relation $y = F(x)$ is suggested. If $F(x)$ can be factored or written in the form

$$F(x) = af(x) + b,$$

where a and b are numerical constants and $f(x)$ contains no constants, the function suggested can be very readily tested graphically. This includes evidently the special cases when $a = 1$ and when $b = 0$; *i.e.*, the functions,

$$y = af(x); \; y = f(x); \text{ and } y = f(x) + b.$$

In all of these cases let $f(x) = z$, and for each value of x of the data compute the corresponding value of z. Construct

a new plot with values of z as abscissæ and the corresponding (observed) values of y as ordinates. The general equation of the new line will then be

$$y = az + b,$$

or that corresponding to the above special cases,

$$y = az;\ y = z;\ \text{and } y = z + b,$$

all of which are equations of a straight line, the constants of which may readily be determined as described above. Whether the assumed equation $y = af(x) + b$ represents the experimental data or not can thus be judged by the magnitude and sign of the deviations of the plotted data from the straight line. If the correct function has been assumed, the values of the constants a and b should be corrected by means of a residual plot, provided the precision of the data warrants it.

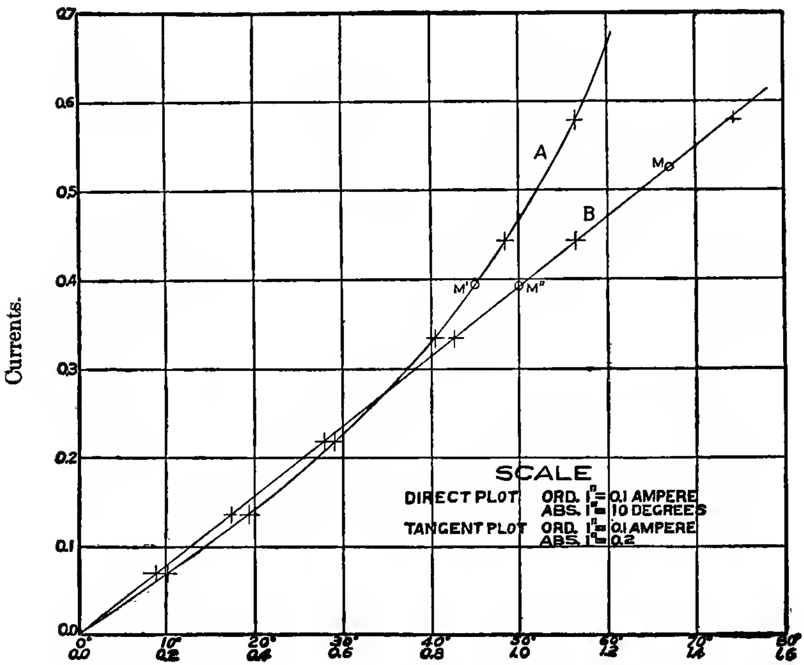
Problem.—Trigonometric Functions. Suppose that with a certain galvanometer the following deflections θ are produced by the currents, I, respectively, and it is desired to determine the law of the galvanometer, *i.e.*, the form of the function $I = F(\theta)$, so that the current corresponding to any deflection may be computed.

OBSERVED DATA.

Deflection θ.	*Current I.*	$z = \tan\theta$.
10.17°	0.0704	0.1794
19.27°	0.1368	0.3496
29.16°	0.2184	0.5580
40.47°	0.3348	0.8532
48.45°	0.4430	1.128
55.90°	0.5780	1.477

The data plotted directly with values of I as ordinates and θ as abscissæ are found to lie along a curve A, Plot II., which evidently suggests the relation $I = a \tan\theta$ where a is constant; for $I = 0$ when $\theta = 0°$ and I approaches a very great

value (infinity), for $\theta = 90°$. Comparing the suggested equation with $y = af(x)$, we see $f(x) = \tan\theta$. Hence, to test the suggested equation, we compute the value $z = \tan\theta$ for each observed value of θ, and construct a new plot with the values of I as ordinates as before, and values of z as



Deflections θ—Curve A.
Tangent θ—Curve B

PLOT II.

abscissæ. The line best representing these data is shown in B. This line must necessarily pass through the origin, since the current and corresponding deflection of the galvanometer approach the value zero simultaneously. The galvanometer is seen to follow the law of tangents between 0° and 60°. Since the line B passes through the origin, the value of the constant a, *i.e.*, the tangent which the line

makes with the axis of X, is readily found from the co-ordinates $x'' \ y''$ of a single point M to be

$$a = \frac{y''}{x''} = \frac{0.525}{1.34} = 0.392.$$

The value of a, in this particular case, can also be obtained as follows: Since $\tan 45° = 1$, it follows from $y = a \tan\theta$ that $y = a$, for $\theta = 45°$ or $z = 1$; *i.e.*, the ordinate of curve A at 45° or the ordinate of curve B at $z = 1$ gives the value of a directly. By this method we find $a = 0.393$ from M', curve A, and $a = 0.392$ from M'', curve B, both values being in good agreement with that obtained in the usual way.

The desired formula for the galvanometer is, therefore,

$$\mathrm{I} = 0.392 \tan \theta.$$

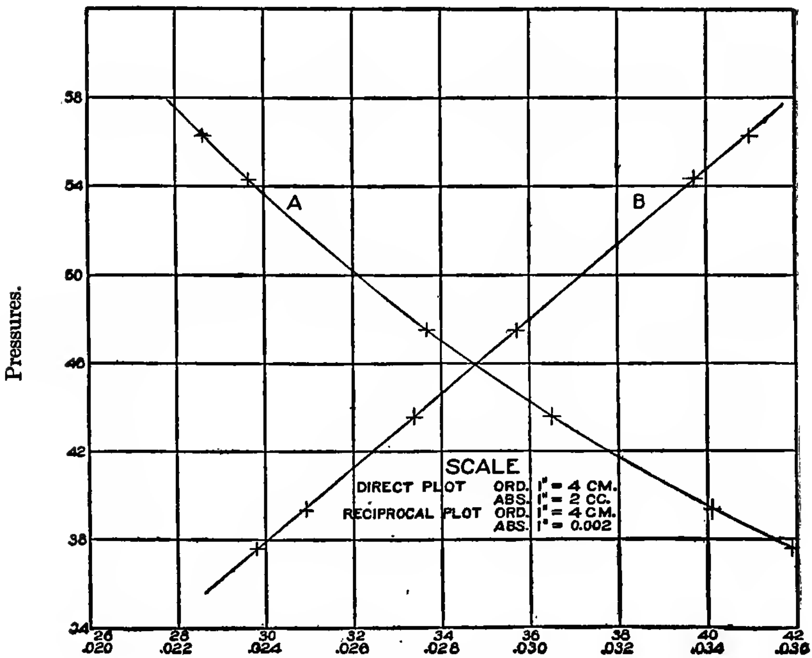
If the observations had been extended beyond $\theta = 60°$ and it were found that the points corresponding to these data regularly deviated from the straight line, the conclusion would be that the instrument followed the law of tangents only within certain limits, which could be thus determined.

Problem.—Reciprocal Functions. Suppose the volume v of a definite mass of gas kept at constant temperature is determined at various pressures p with the following results, and it is desired to find the law connecting p and v; *e.g.*, to determine the form of the function $p = f(v)$.

Pressure p in cm. of Hg.	*Volume v in c. c.*	$\frac{1}{v}$
37.60	41.90	0.02380
39.35	40.13	0.02493
43.59	36.51	0.02739
47.50	33.67	0.02971
54.34	29.65	0.03373
56.26	28.63	0.03497
58.28	27.70	0.03610

Constructing a direct plot from these data, we obtain a slightly curved line A, Plot III. The volume diminishes as the pressure increases, but not proportionally, since the data do not lie along a straight line. The curve suggests an equi-

lateral hyperbola referred to its asymptotes as axes, the equation of which is $xy = \text{const.}$ If this suggested relation be the correct one, *i.e.*, if $vp = \text{const.}$, or, otherwise written, $p = \text{const.} \left(\frac{1}{v}\right)$, by changing the variable from v to $z = \frac{1}{v}$, the resulting equation becomes $p = \text{const. } z$, the equation of



Curve A,—Volumes.
Curve B,—Reciprocal of Volumes

PLOT III.

a straight line. Constructing, therefore, a second plot, with the same values of p as ordinates and the reciprocal values of v as abscissæ, we obtain curve B, which should be a straight line if the gas in question follows Boyle's law within the errors of the experiment. This is seen to be the case. If it were not the case, a study of the deviations of the data

from the straight line would afford a proper means of discussion of the deviations of the gas from Boyle's law.

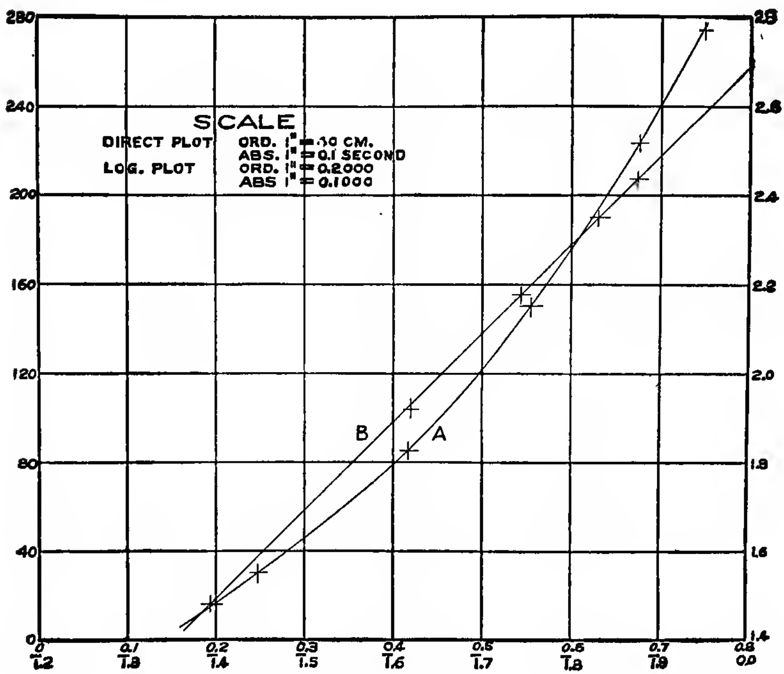
Another method of treating this problem would be to compute the product pv for each pair of values of p and v and then to discuss the values of the product graphically as follows. With values of p as abscissæ construct a plot with corresponding values of pv as ordinates. If the data satisfy the relation $pv = \text{const.}$ within the experimental error, the best representative line will be parallel to the axis of abscissæ, with the values of pv distributed alternately and about equally on either side. If, on the other hand, the gas deviates from Boyle's law, as many gases do even under ordinary conditions, and as all gases do at very great values of p, the resulting curve will give information not only as to the amount of the deviations, but also as to the method of correcting the assumed simple relationship to make it better conform with experimental facts.

The Logarithmic Method. Exponential Functions.—If the data of a direct plot are found to deviate continually from a straight line, they may very often be represented by an exponential equation of the form $y = mx^n$ where m and n are constants which may have any value. Cases of this kind are of very frequent occurrence, and it is, therefore, of great importance to be able to test this relationship and to determine the numerical values of the constants m and n. This can always be done by means of a *Logarithmic Plot; i.e.*, a plot constructed with the values of the logarithms of y as ordinates and the corresponding logarithms of x as abscissæ. For, if we take logarithms of both sides of the equation $y = mx^n$, we have

$$\log y = n \log x + \log m.$$

Changing the independent variables x and y in this equation to x' and y' respectively, by putting $x' = \log x$ and $y' = \log y$, and writing $b = \log m$, the equation becomes $y' = nx' + b$. This is the equation of a straight line of which the intercept on the axis of Y is $b = \log m$, and of which the

natural tangent of the undistorted angle which it makes with the axis of X is n. Hence the constants in the original equation $y = mx^n$ may be obtained at once by looking out the number m whose logarithm, b, is the intercept of the straight line on the axis of Y, and by determining the tangent which the line makes with the axis of X.



PLOT IV.

Here, again, the values of the constants m and n as thus determined are usually reliable to not more than 0.5%, and hence, if the orginal data warrant it, they should be further corrected by means of a residual plot.

Problem.—The logarithmic method will now be illustrated by discussing data obtained for a body falling freely under the influence of gravity. Suppose experiments gave the following values for the distance s, through which a ball fell in the time t, and it is desired to deduce the law between s

and t; *i.e.*, to find the equation by which the distance s can be computed for any value of t.

OBSERVED DATA.

Distance s in centimeters.	*Time t in seconds.*	$s' = \log s.$	$t' = \log t.$
30.13	0.2477	1.4790	$\bar{1}.3939$
85.26	0.4175	1.9308	$\bar{1}.6207$
150.39	0.5533	2.1772	$\bar{1}.7430$
223.60	0.6760	2.3495	$\bar{1}.8300$
274.20	0.7477	2.4381	$\bar{1}.8737$

A direct plot A, Plot IV., of s and t, shows at once that s and t are not proportional. The regular deviation from a straight line suggests an exponential curve, *i.e.*, $s = mt^n$. To test this relation, we construct on ordinary co-ordinate paper a "*logarithmic plot*" B with $s' = \log s$ as ordinates and $t' = \log t$ as abscissæ. Convenient scales which distribute these values about 45° across the paper are $1'' = 0.2$ for the ordinates and $1'' = 0.1$ for the abscissæ. It is to be noticed that the values of t, being less than unity, lead to values of $\log t$ with negative characteristics. The abscissæ are, therefore, laid off to the left of the origin as indicated, the plot thereby lying in the second quadrant. The data are seen to lie very closely along a straight line, the constants of which are to be determined as described on page 46. Thus the intercept of the line on the axis of Y is $b = \log m = 2.688$, whence $m = 488$. The tangent which the line makes with the axis of X is found to be $n = 1.995$ or $n = 2.00$ within the error of plotting.

The desired equation is, therefore, $s = 488\ t^{2.00}$. Since the law of falling bodies is known to be $s = \frac{1}{2}gt^2$, it follows that $\frac{1}{2}g = 488$, or the mean value of g from the data, within the error of direct plotting, is $g = 2 \times 488 = 976 \dfrac{\text{cm.}}{\text{sec.}^2}$.

If the data s and t are reliable to more than about 0.5 per cent., the constants m and n should be corrected by means of a residual plot.

Attention should be called to one important point in this

connection. In constructing a plot like the above in which the intercept on the axis Y is to be determined, it is convenient to choose the units in which the abscissæ are expressed such that the resulting line cuts the axis of Y without a long extrapolation. By a suitable choice of units this condition can always be attained, for increasing or diminishing the unit expressing the abscissæ by a multiple of ten does not affect the slope of the line, but simply shifts it parallel with itself to or from the origin.

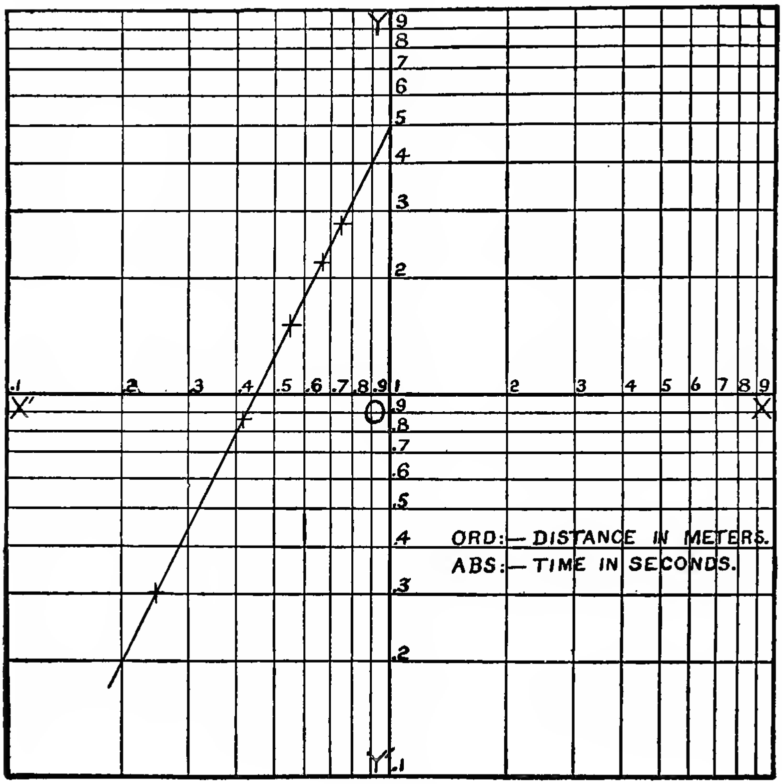
Logarithmic Plotting-paper.—When the constants of a number of exponential curves of the type $y = mx^n$ are to be determined, a great saving of time and labor may be effected by using so-called logarithmic co-ordinate paper. Four quadrants of such paper are shown in Plot V. The length OX is laid off equal to OY and put equal to 10 or some integral power of 10 units. This is then subdivided into spaces such that the distances 1–2, 1–3, 1–4, etc., are proportional to the logarithms of 2, 3, 4, etc. Thus the point numbered 2 is located not at two-tenths the distance from O to X, as in ordinary plotting-paper, but at $\log 2 = 0.301$ of the distance OX. The rulings thus become more and more crowded together as they proceed from O to X and Y. Continuing the rulings beyond 10 in either direction, it is evident that the unit square XOY repeats itself indefinitely, since the value of the logarithm of any quantity multiplied by 10^k, where k is a positive or negative integer, is equal to the logarithm of the original quantity plus k. Thus the point marked 0.2 is laid off at a distance equal to $\log 0.2 = \log (2 \times 10^{-1}) = \log 2 - 1 = -1 + 0.301$ to the left of O; *i.e.*, just the distance OX to the left of the point marked 2, etc.

It is evident that a series of values of x and y which satisfy the equation

$$y = mx^n$$

will, if plotted *directly* on logarithmic paper, lie along a straight line; for the paper has the effect of locating the data y and x at points proportional to their logarithms, and

it has been shown on page 52 that this leads to a straight line. We are thus saved the labor of looking out logarithms and locating them on rectangular co-ordinate paper as previously explained. Moreover, since the scales of ordinates and abscissæ are here necessarily equal, the slope of the



PLOT V.

resulting line is undistorted, and, therefore, the tangent which it makes with the axis of X is obtained by measuring off with a scale the distance $y'' - y'$ and $x'' - x'$ of two points on the line and taking their ratio, $\frac{y'' - y'}{x'' - x'} = n$. This gives the desired value of the exponent of x.

To find the value of the constant m, we note that in the equation

$$\log y = n \log x + \log m$$

when $x = 1$,

$$\log y = \log m.$$

Hence, since values read off on logarithmic paper correspond to the numbers of which the spacings are the logarithms, the intersection of the logarithmic plot with the ordinate through $x = 1$ gives at once the value of m, whereas with rectangular plotting paper the intercept with the Y-axis gives the value of $\log m$.

The following difference between logarithmic plots, drawn on rectangular and on logarithmic paper, should also be noted. Suppose the line representing the data plotted on logarithmic paper intersects the ordinate not through $x = 1$, but through $x = 10^k$, where k is any positive or negative integer. Let m' be the value of the intercept. To obtain the value of m in the equation $y = mx^n$ from m', we have $y = m'$ for $x = 10^k$, *i.e.*,

$$m' = m\,10^{kn}$$

$$\text{or } m = \frac{m'}{10^{kn}}.$$

In the case of rectangular paper, on the other hand, the line intersects the ordinate through $x = \log 10^k = k$ at m''.

To obtain m from this intercept, we have

$$\begin{aligned} m'' = \log y &= nk \log 10 + \log m \\ &= nk + \log m \end{aligned}$$

Hence $\log m = m'' - nk$.

The solution of the problem discussed on page 54, by the use of logarithmic paper is shown on a reduced scale in Plot V. Values of s and t are plotted directly and lie, as is seen, along a straight line. It is convenient to express here values of s in meters instead of centimeters. The tangent which this line makes with the axis of X measured off directly along OY and OX' is found to be 2.00. Extrapolating the

line to cut the ordinate OY through $x = 1$, we see the intercept to be

$$m = 4.89.$$

The equation connecting s and t is therefore

$$s = 4.89\, t^{2.00}$$

which agrees, within the error of plotting, with that previously obtained with rectangular co-ordinate paper, when we remember that in the above equation s is expressed in meters instead of centimeters.

Equations of the Form $y = m(x + \beta)^n$.

A slightly more complicated relation than that represented by the equation $y = mx^n$, which may also be treated by the logarithmic method, is that represented by the formula

$$y = m(x + \beta)^n$$

where β is a constant. If a logarithmic plot be made with data x_1y_1, x_2y_2, etc., which satisfy an equation of this form, the points will not lie along a straight line for both large as well as small values of the variables. Suppose it is found that for large values of x and y the curve is practically straight, but for small values it becomes curved. Under these circumstances it is worth while to see if the best representative logarithmic plot cannot be rectified into a straight line by assuming an equation of the above form, for by putting $z = x + \beta$ the equation reduces to

$$y = mz^n$$

from which m and n are easily determined, if values of y and z are plotted logarithmically. It is only necessary therefore to find the value of the constant β to be added to all values of x. This may be found as follows. Select two points x_1y_1 and x_2y_2 near the ends of the original logarithmic plot. Compute the ordinate y_3 of an intermediate point such that

$$\log y_3 = \tfrac{1}{2} \log y_1 + \tfrac{1}{2} \log y_2$$

$$\text{or } y_3 = \sqrt{y_1 y_2}$$

and look out its corresponding abscissa x_3 on the line.

Then it follows, if the data satisfy an equation of the form $y = m(x + \beta)^n$, that

$$\log(x_3 + \beta) = \tfrac{1}{2}\log(x_1 + \beta) + \tfrac{1}{2}\log(x_2 + \beta)$$

$$\text{or } x_3 + \beta = \sqrt{(x_1 + \beta)(x_2 + \beta)}$$

$$\text{from which } \beta = \frac{x_3^2 - x_1x_2}{x_1 + x_2 - 2x_3}.$$

Having thus obtained β, proceed in the usual manner to determine m and n from a new logarithmic plot of the equation $y = mz^n$ where $z = x + \beta$.

Equations of the Form $y = m\,10^{nx}$; $y = m\,\epsilon^{nx}$

Data satisfying equations of the form

$$y = m\,10^{nx} \text{ and } y = m\,\epsilon^{nx},$$

where ϵ is the base of Naperian logarithms, may also be treated graphically by the following special logarithmic method. Taking logarithms of these equations, we obtain

$$\log y = nx + \log m$$

$$\text{and} \qquad \log y = Mnx + \log m$$

respectively, where $M = 0.4343$ is the modulus for reducing Naperian to common logarithms. If, on ordinary plotting paper, values of $y' = \log y$ are plotted as ordinates and the unchanged values of x as abscissæ, the resulting curves will be straight lines; the intercept on the axis of Y for $x = 0$ will give $y' = \log m$ and the tangents of the lines with the axis of X will give the values of n and Mn respectively.

Precision of Plotting.—The question now arises as to the precision of the constants deduced from a direct or rectified plot. In discussing this question, we will consider only errors inherent in the process of plotting and interpolation, and in the plotting-paper itself.

The error of estimating tenths of the smallest division, together with the uncertainty introduced by the width of the lines locating the data and the inaccuracies in the paper due to errors in ruling and unequal shrinkage, make 0.02 inch a fair estimate of the extreme precision of reading or

plotting. If the plot be 10 inches on a side (about the maximum size ordinarily employed), the fractional precision attainable cannot therefore be greater than about $\frac{0.02}{10} = 0.002$, or 0.2 per cent. A more probable estimate of the precision ordinarily attained in direct plots is 0.4 to 0.5 per cent. Constants deduced from a direct plot of the size considered can therefore be relied upon only to this degree of precision; that is, in general, to three significant figures, with the fourth doubtful.

If the experimental data are reliable to four or more significant figures (*i.e.*, to 0.1 per cent. or better), some of the precision will evidently be sacrificed in the direct plot unless a much larger plot be made. In order that the full precision of such data may be utilized, the direct plot should be followed by a so-called *residual plot,* by means of which the constants first obtained can be corrected and rendered more precise. By this procedure the precision of the graphical method may be greatly extended. The procedure to be followed in constructing a residual plot will now be considered.

Residual Plot.—A residual plot is one in which the deviations of the observed data from the "best representative line" are plotted on an enlarged scale. It serves to correct the position of this line among the points, to correct the numerical value of the constants, and to test whether the data follows the assumed law within the precision of the measurements. It is constructed as follows. Substitute in the equation $y = ax + b$, deduced for the best representative line, which may be either a direct or rectified plot, the observed values of x, and compute the corresponding values of y. The differences between these computed values of y and the corresponding observed values are called the *residuals.* A study of the sign and magnitude of these residuals furnishes much valuable information regarding the representative character of the "best line" chosen, and the graphical discussion of these constitutes the residual plot. If a plot be made (preferably on the same paper and with the

same scale of abscissæ as the straight line plot), with the values of the residuals $r = y$ (observed) — y (computed) as ordinates, and the corresponding values of x as abscissæ, we obtain a graphical representation of the deviations of the observed data from the line assumed to best represent them. To better study these deviations, they should be plotted on a large scale. In effect, the process is to project the "best representative line" horizontally and to magnify the deviations of the plotted data from it. If it is found that the plotted residuals lie alternately and about equal distances on either side of the horizontal line passing through the zero of the residuals, the conclusion is that the original line is the best line which can be drawn to represent the data. In general, however, it will be found that a new line can be drawn among the residuals which will distribute them more nearly alternately on either side. The values of the tangent a and intercept b, found for the original representative line, should therefore be corrected by the values of the tangent and intercept respectively of the new best representative line of the residual plot, read off of course on the scales on which it is plotted. In this way the original constants may be corrected to the fourth significant figure. It is sometimes necessary to follow the first by a second residual plot when extreme precision is desired.

If the residuals are found to deviate systematically from the straight line, the conclusion is that the data cannot be represented by the line in question within the precision of the measurements. In such a case a new formula should be sought.

Illustration of a Residual Plot. The procedure to be followed in making a residual curve or plot will be illustrated by the data given in the Problem discussed in Plot I., p. 45. The equation of the best representative straight line for these data was found to be

$$r = 0.0278\,t + 10.13.$$

To test whether this equation is the best which can be obtained to represent the given data, we proceed to compute

the residuals, as described above, by substituting the observed values of t and computing r'.

t observed.	r observed.	r' computed.	$r-r'$ first residuals.	r'' computed.	$r-r''$ second residuals.
10.50	10.421	10.422	—0.001	10.413	+0.008
29.49	10.939	10.949	—0.010	10.946	—0.007
42.70	11.321	11.317	+0.004	11.317	+0.004
60.01	11.799	11.798	+0.001	11.802	—0.003
75.51	12.242	12.229	+0.013	12.237	+0.005
91.05	12.668	12.661	+0.007	12.673	—0.005
			$\Sigma\,(r-r')^2 = 437$		$\Sigma\,(r-r'')^2 = 188$

Inspection of these residuals, column 4, affords valuable information, but they can be better studied graphically, especially if the number of observations is great. The scale to be chosen for the ordinates should not be greater than about 1 inch to 0.01 ohm, since this will permit the residuals to be plotted directly without interpolation to the last place of significant figures of the data, while by estimation the plot can be read to the next place of figures; *i.e.*, to 0.0001 ohm, which is more than ten times the precision of the data. The plot may conveniently be made on the same sheet as the direct plot, using the same scale of abscissæ as shown in Plot I., p. 45. The heavy horizontal line through O represents the line $A'A''$ projected horizontally. The residuals, plotted on a magnified scale, are connected by dotted lines. Inspection shows that the positive residuals preponderate, and that a new line $B'B''$ can be drawn which will distribute the residuals more nearly alternately on either side of it. The original line $A'A''$ should evidently have been drawn with a slightly greater inclination. The value of the intercept of the new line $B'B''$ on the axis of Y (on the scale of residuals) is—0.011. The tangent of the angle which it makes with the axis of X is obtained from the ordinates and abscissæ of two points B' and B'' on the line respectively: thus $x' = 5.00°$, $y' = 0.0095$; and $x'' = 95.05°$, $y'' = 0.0135$. Hence

$$\frac{y''-y'}{x''-x'} = \frac{0.0135-(-0.0095)}{95.05-5.00} = 0.00025.$$

Hence the constants of the original equation should be corrected by these amounts, thus becoming

$$b' = 10.13 - 0.011 = 10.119$$
$$a' = 0.0278 + 0.00025 = 0.02805.$$

The corrected equation connecting r and t is, therefore,

$$r = 0.02805\ t + 10.119.$$

This represents the original data much better than the first equation obtained, as may be seen from the sign and magnitude of the new set of residuals $r - r''$ computed from the corrected equation and given in the last column of the table. There is now seen to be no systematic deviation among the residuals, and the sum of their *squares* is seen to be much less than in the case of the residuals from the first equation.

Interpolation Formulæ.—It frequently happens that experimental data whose locus differs slightly but progressively from a straight line cannot be represented by a two constant formula of the general exponential form $y = mx^n$. This is the case, for example, with data on the coefficient of expansion of many substances over wide ranges of temperature. To obtain an algebraic relation for such cases, interpolation formulæ of the general form

$$y = a + bx + cx^2 + dx^3 + \ldots$$

are usually assumed. The number of terms to be taken in this equation (*i.e.*, the number of constants to be determined) depends upon the precision of the data and on the extent of the deviation of the curve representing them from a straight line.

The values of the constants in such an equation are in general best determined analytically. For this purpose it is necessary to know at least as many pairs of values of x and y as there are constants to be determined. Thus, if the equation assumed to represent the data be $y = a + bx + cx^2$, it is necessary to know at least three pairs of values of x and y which, substituted in the equation, will lead to three simultaneous equations, from which the values of the three unknown constants, a, b, c, can be at once determined by

elimination. It is seldom necessary to carry the series beyond the fourth term dx^3; in fact, three terms are sufficient for most purposes.

In general, however, the experimental data furnish many more pairs of values of x and y than there are constants to be determined. In all such cases the most probable value of the constants can be determined by the graphical procedure described below or by the method of Least Squares.

Graphical Solution.—Let x_1y_1, x_2y_2, . . . x_ny_n, be the numerical values of pairs of observations on the variables x and y, which are assumed to satisfy the equation

$$y = a + bx + cx^2.$$

Any three pairs of values substituted in this equation will give three simultaneous equations from which a, b, and c can be computed, but the values of these constants will vary to a certain extent according to which sets of values of x and y are chosen. The simplest procedure by which to obtain the *best* or most probable values of a, b, and c, is to plot all values of x and y and draw the best representative line among them. Then select three points *on this line,*—one near each end and one half-way between for convenience,—determine their ordinates and abscissæ, and with these three pairs of values form three simultaneous equations and compute a, b, and c. Having obtained the constants in this manner, they may be further corrected by computing residuals and studying these by means of a residual plot, although this requires both care and judgment. A more exact although more laborious method of procedure is the analytical solution of the equation by the method of Least Squares.

Least Square Solution.—As before, let x_1y_1, x_2y_2, . . . x_ny_n, be numerical values of the observations, and

$$y = a + bx + cx^2$$

the equation the constants of which are to be determined. This may be written

$$y - a - bx - cx^2 = 0.$$

If the observed values x, y, were free from all experimental errors and the equation represented the law connecting them, each pair would exactly satisfy the equation, with proper numerical values of the constants. This, however, is not the case, since all observations are liable to indeterminate error. Hence, if the observations be substituted in the equation, the right member will not in general equal zero, but will differ from zero by some small quantity v called the residual error, which may be plus or minus. Thus, by substituting the observations in the assumed equation, we get the following so-called "*observation equations*":—

$$\begin{aligned} y_1 - a - bx_1 - cx_1^2 &= v_1, \\ y_2 - a - bx_2 - cx_2^2 &= v_2, \\ \cdots\cdots&\cdots\cdots \\ \cdots\cdots&\cdots\cdots \\ y_n - a - bx_n - cx_n^2 &= v_n, \end{aligned}$$

from which the most probable values of the constants a, b, c, are to be determined. By the principle of Least Squares those values of a, b, c, are the most probable which make the sum of the squares of the residual errors v a minimum; *i.e.*, those which make the value of $\Sigma v^2 = v_1^2 + v_2^2 + \ldots v_n^2$ a minimum. The expression Σv^2 is a function of the quantities a, b, c, and the condition that it shall be a minimum is that its first differential coefficient with respect to these variables shall be zero, and its second differential coefficient positive. The latter test need not be applied, however, as inspection will distinguish between maximum and minimum values, the limit of the former being evidently infinity.

Applying this condition to the above observation equations, we have

$$\frac{d(\Sigma v^2)}{da} = \frac{d}{da}(v_1^2 + v_2^2 + \cdots + v_n^2) =$$

$$2\left(v_1\frac{dv_1}{da} + v_2\frac{dv_2}{da} + \cdots + v_n\frac{dv_n}{da}\right) = 0.$$

$$\frac{d(\Sigma v^2)}{db}=\frac{d}{db}(v_1^2+v_2^2+\cdots+v_n^2)=$$

$$2\left(v_1\frac{dv_1}{db}+v_2\frac{dv_2}{db}+\cdots+v_n\frac{dv_n}{db}\right)=0.$$

$$\frac{d(\Sigma v^2)}{dc}=\frac{d}{dc}(v_1^2+v_2^2+\cdots+v_n^2)=$$

$$2\left(v_1\frac{dv_1}{dc}+v_2\frac{dv_2}{dc}+\cdots+v_n\frac{dv_n}{dc}\right)=0.$$

Substituting the values of v_1, v_2, etc., and differentiating, we obtain

$$(y_1-a-bx_1-cx_1^2)+(y_2-a-bx_2-cx_2^2)+\cdots+(y_n-a-bx_n-cx_n^2)=0,$$

$$(y_1-a-bx_1-cx_1^2)\,x_1+(y_2-a-bx_2-cx_2^2)\,x_2+\cdots+(y_n-a-bx_n-cx_n^2)\,x_n=0,$$

$$(y_1-a-bx_1-cx_1^2)\,x_1^2+(y_2-a-bx_2-cx_2^2)\,x_2^2+\cdots+(y_n-a-bx_n-cx_n^2)\,x_n^2=0,$$

which may be simplified to the equations

$$\Sigma y-\Sigma a-b\Sigma x-c\Sigma x^2=0.$$
$$\Sigma xy-a\Sigma x-b\Sigma x^2-c\Sigma x^3=0.$$
$$\Sigma yx^2-a\Sigma x^2-b\Sigma x^3-c\Sigma x^4=0.$$

These are called the "normal equations," from which the values of the constants a, b, c, may be computed by the ordinary methods of elimination, there being now the same number of equations as unknowns. It will readily be seen that the process of substituting the values of Σy, Σxy, Σxy^2, etc., in the normal equations and the subsequent solution of the equations for the constants is a tedious process, the labor involved increasing rapidly with the number of constants to be determined.

For further details regarding the method of Least Squares consult Bartlett's *The Method of Least Squares*, Wright's *Treatise on the Adjustment of Observations*, or Merriman's *Least Squares*. For special Graphical Methods see Peddle's *The Construction of Graphical Charts*.

PART III.

SOLUTION OF ILLUSTRATIVE PROBLEMS,

AND

PROBLEMS.

SOLUTION OF ILLUSTRATIVE PROBLEMS.

Before proceeding to the numerical solution of a precision problem, the student should first decide the following questions:—

First.—Is the formula to be discussed in the simplest form for precision treatment? It frequently happens, by the omission of certain terms the deviations in which evidently produce a negligible effect on the final result, that an apparently complex formula can be reduced to a more convenient form. If it can be reduced to a product function, this should always be done.

Second.—From a consideration of the form of function to be discussed, a decision should be made as to which method it is better to employ in the solution; that is, whether to use the general "deviation method" involving differentiation of the function or the fractional or "inspection" method.

Third.—Having decided these questions, the *statement* of the problem should be studied; that is, all given data should be systematically written down and inspected to see if they are in the proper form for applying the method of solution decided upon. If this is not the case, numerical deviations δ or Δ should be changed over into their corresponding fractional deviations $\frac{\delta}{m}$ or $\frac{\Delta}{M}$, or *vice versa*, as the case may be. Only after the problem has been consistently stated should the actual solution be begun.

These general directions are illustrated below by the solution of several typical problems.

Problem 1.—Given the following mean values of the weight of four substances with their respective deviation measures:—

$w_1 = 3147.226$ gms.	$A.D. = 0.312$ gm.
$w_2 = 100.4211$ gms.	reliable to 0.015 per cent.
$w_3 = 1.3246$ gms.	Probable error $P.E. = 0.0011$ gm.
$w_4 = 604.279$ gms.	reliable to 1 part in 5000.

(*a*) Indicate any superfluous figures in the above measurements, considering each independent of the others.

Each quantity should be carried out to two places of uncertain figures as indicated by the two significant figures in its average deviation (Rule III., p. 24). The average deviation of each measurement should therefore be computed for each measurement if it is not already given. Computing the average deviations and applying Rules I., II., and III., it will be seen that the correct number of figures to be retained is as follows:—

$w_1 = 3147.23$ gms. $A.D. = \delta_1 = 0.31$ gm.

$w_2 = 100.421$ gms. $A.D. = \delta_2 = 0.015$ gm. as $100\dfrac{A.D.}{100} = 0.015.$

$w_3 = 1.3246$ gms. $A.D. = \delta_3 = 0.0013$ gm. as $P.E. = 0.85\ A.D.$

$w_4 = 604.28$ gms. $A.D. = \delta_4 = 0.12$ gm. as $\dfrac{A.D.}{600} = \dfrac{1}{5000}.$

(*b*) Which is the most and which the least precise of these measurements?

When the quantities whose precision is to be compared are not of approximately equal magnitude, their relative precision is found by comparing their fractional or percentage deviations, but not their average deviations or probable errors. Hence with the above data we must compute the fractional or percentage deviation of each of the quantities.

For w_1, $100\dfrac{\delta_1}{w_1} = 100\dfrac{0.31}{3100} = 0.010$ per cent.;

w_2, $100\dfrac{\delta_2}{w_2} = 100\dfrac{0.015}{100} = 0.015$ per cent.;

w_3, $100\dfrac{\delta_3}{w_3} = 100\dfrac{0.0013}{1.3} = 0.10$ per cent.;

w_4, $100\dfrac{\delta_4}{w_4} = 100\dfrac{0.12}{600} = 0.020$ per cent.

Therefore, the order of precision is w_1, w_2, w_4, w_3. It is to be noted that, although w_3 is weighed to a much smaller fraction

of a gram than any of the other quantities, it has by far the largest percentage deviation, and is to be regarded, therefore, as the least precise measurement.

(*c*) Find the sum of the measurements and its deviation measure, retaining the proper number of significant figures in the computation.

$$M = w_1 + w_2 + w_3 + w_4.$$

The quantity having the largest *A.D.* is w_1, its average deviation being 0.31 gm. In the units chosen to express the measurements, the first and second decimal places are uncertain. Therefore, by Rule IV., page 24, two decimal places only should be retained in each of the other quantities to be added.

$$\begin{array}{rr} & 3147.23 \text{ gms.} \\ & 100.42 \text{ gms.} \\ & 1.32 \text{ gms.} \\ & \underline{604.28 \text{ gms.}} \\ M = & 3853.25 \text{ gms.} \end{array}$$

The resultant deviation Δ of the sum M is

$$\Delta = \sqrt{\Delta_1^2 + \Delta_2^2 + \Delta_3^2 + \Delta_4^2}.$$

$$\Delta_1 = \frac{\partial M}{\partial w_1} \cdot \delta_1 = \delta_1 = 0.31 \text{ gm.}$$

Similarly, $\Delta_2 = \delta_2 = 0.015$ gm., which is negligible.

$\Delta_3 = \delta_3 = 0.0013$ gm., which is negligible.

$\Delta_4 = \delta_4 = 0.12$ gm.

Therefore, $\Delta = \sqrt{\delta_1^2 + \delta_4^2} = \sqrt{0.31^2 + 0.12^2} = 0.33$ gm.

(*d*) Find the product of the four quantities and its precision measure.

$$M = w_1 \,.\, w_2 \,.\, w_3 \,.\, w_4.$$

By Rule V., page 24, the least precise factor is w_3, which is good to only 0.10 per cent., and in which, therefore, five significant figures should properly be retained in a computation, the last two being uncertain. Five figures should likewise be retained in each of the other factors, and five place logarithms should be used in the computation.

$$\begin{array}{ll} w_1 = 3147.2 & \log = 3.49793 \\ w_2 = 100.42 & \log = 2.00182 \\ w_3 = 1.3246 & \log = 0.12209 \\ w_4 = 604.28 & \log = \underline{2.78124} \\ & \log M = 8.40308 \end{array}$$

or $M = 252980000.$ gms.[4]

The precision of M should be computed by the fractional or inspection method, as it is a product function. By referring back to problem (b), it will be seen that the percentage precision of w_1, w_2, and w_4 is between five and ten times as great as that of w_3. Hence practically all of the uncertainty in the product will result from the deviation in this factor alone. As M is directly proportioned to the first power of w_3,

$$100\frac{\Delta_3}{M} = 100\frac{\delta_3}{w_3}$$

and therefore the percentage deviation of the product $100\frac{\Delta}{M} = 0.10$ per cent. Hence $\Delta = M \times \frac{0.10}{100} = 250000.\ \overline{\text{gms.}}^4$

(e) Suppose the quantities are to be combined by the formula

$$M = w_1 \times w_2 - w_3 \times w_4.$$

Compute M and its deviation measure.

Before substituting the values of w in the actual calculation of M, it is always well to note the approximate value of the terms involved. From inspection of the data it is evident that $w_1 \times w_2 = 310000.$ approximately, and, as the least precise factor w_2 is good to 0.015 per cent., this product should be computed to six significant figures, of which the last two will be uncertain. The second place of uncertain figures thus falls in the units' place; anything beyond this is, therefore, negligible. The term $w_3 \times w_4 = 600.$ approximately, and, as this is to be subtracted from $w_1 \times w_2$, it is useless to compute it beyond the units' place, *i.e.*, three significant figures are sufficient. Six place logarithms should therefore be used in computing $w_1 \times w_2$, and three place logarithms, short multiplication, or a slide rule in computing $w_3 \times w_4$.

$$\begin{array}{ll}
\text{Thus } w_1 = 3147.23 \text{ gms.} & \log w_1 = 3.4979284 \\
w_2 = 100.421 \text{ gms.} & \log w_2 = \underline{2.0018245} \\
 & \log w_1 \times w_2 = 5.4997529 \\
 & \therefore w_1 \times w_2 = 316048.\ \overline{\text{gms.}}^2 \\
w_3 = 1.32 \text{ gms.} & \log w_3 = 0.121 \\
w_4 = 604. \text{ gms.} & \log w_4 = \underline{2.781} \\
 & \log w_3 \times w_4 = 2.902
\end{array}$$

$$\therefore w_3 \times w_4 = 798.\ \overline{\text{gms.}}^2$$

$$M = 316048\ \overline{\text{gms.}}^2 - 798\ \overline{\text{gms.}}^2 = 315250\ \overline{\text{gms.}}^2$$

To determine the precision of M, we note that, if the formula be treated in the form $M = w_1 \times w_2 - w_3 \times w_4$, we must use the general differential method and find the effect of each δ on M, and take the square root of the sums of the squares. It is evident, however, since w_3 is good to 0.10 per cent., and w_4 to 0.02 per cent., that the first three significant figures of the product $w_3 \,.\, w_4$ are known exactly, and, therefore, the deviations in w_3 and w_4 introduce no uncertainty in the final result M. The whole uncertainty comes from the measurements w_1 and w_2. As $w_3 \times w_4$ is also numerically small compared with $w_1 \times w_2$ we may, in the precision discussion, neglect it and write

$$M = w_1 \times w_2 \text{ approximately,}$$

and obtain the precision of M by the fractional method. The resultant fractional deviation in M is

$$\frac{\Delta}{M} = \sqrt{\left(\frac{\Delta_1}{M}\right)^2 + \left(\frac{\Delta_2}{M}\right)^2}$$

But $$\frac{\Delta_1}{M} = \frac{\delta_1}{w_1} = 0.00010$$

and $$\frac{\Delta_2}{M} = \frac{\delta_2}{w_2} = 0.00015$$

Therefore, $$\frac{\Delta}{M} = 0.0001\sqrt{1^2 + \overline{1.5}^2} = 0.00018$$

or $$100\,\frac{\Delta}{M} = 0.018 \text{ per cent.}$$

and $$\Delta = 320000.\ \overline{\text{gms.}}^2 \times 0.00018 = 58\ \overline{\text{gms.}}^2;$$

that is, the value of M is uncertain by $\pm$ 58 units.

The above problem illustrates the manner in which the number of significant figures to be retained in a measurement depends entirely upon the way in which it enters into the computation. Thus in (*d*) w_3 was required to its full precision, while in (*e*) it might have been measured much less precisely.

Problem 2.—It is desired to determine the amount of heat H generated in one hour by a certain incandescent lamp, together with its deviation expressed in calories and in per cent. Suppose mean measurements obtained by an ammeter and voltmeter give $I = 2.501 \pm 0.012$ amperes, and $E = 109.72 \pm 0.34$ volts respectively, and the time of opening and closing the circuit is uncertain by

± 0.5 second at each operation. The value of H expressed in calories is

$$H = 0.2390\ I.E.t.$$

The expression for H is, as it stands, a simple product function of the variables I, E, and t, and cannot be further simplified. The problem should therefore be solved by the fractional method.

The data given are

$I = 2.501$ amp. $\delta_I = 0.012$ amp.

$E = 109.72$ volts. $\delta_E = 0.34$ volt.

$t = t_2 - t_1 = 1$ hour $= 3600$ sec.

$\delta_{t_1} = \delta_{t_2} = 0.50$ sec., but δ_t is unknown.

The desired results are the value of H, its deviation Δ in calories, and its percentage deviation $100\,\frac{\Delta}{H}$.

The first step is to change the given deviations δ in each component into their respective fractional deviations.

The fractional deviation in I is $\frac{\delta_I}{I} = \frac{0.012}{2.5} = 0.0048.$

The fractional deviation in E is $\frac{\delta_E}{E} = \frac{0.34}{110.} = 0.0031.$

To find the fractional deviation in t, we must first compute its numerical deviation. Since $t = t_2 - t_1$ and $\delta_{t_1} = \delta_{t_2} = 0.5$ sec., the numerical deviation in t is $\delta = \sqrt{\Delta_1^2 + \Delta_2^2}$;

but $\Delta_1 = \frac{\partial t}{\partial t_1} \cdot \delta_{t_1} = \delta_{t_1} = 0.50$ sec.

and $\Delta_2 = \frac{\partial t}{\partial t_2} \cdot \delta_{t_2} = \delta_{t_2} = 0.50$ sec.

Therefore $\delta = \sqrt{0.5^2 + 0.5^2} = 0.70$ sec.,

and the fractional deviation in t is

$$\frac{\delta}{t} = \frac{0.70}{3600} = 0.00019.$$

This is seen to be negligible compared with the fractional deviation in the current I and voltage E (see page 31). Hence the resultant fractional deviation in H is

$$\frac{\Delta}{H} = \sqrt{\left(\frac{\Delta_I}{H}\right)^2 + \left(\frac{\Delta_E}{H}\right)^2}$$

By inspection of the formula for H, since I and E both enter as first power factors,

$$\frac{\Delta_I}{H} = \frac{\delta_I}{I} = 0.0048$$

$$\frac{\Delta_E}{H} = \frac{\delta_E}{E} = 0.0031.$$

Therefore, $\frac{\Delta}{H} = \sqrt{\overline{0.0048}^2 + \overline{0.0031}^2} = 0.0057,$

or the percentage deviation in H is $100\frac{\Delta}{H} = 0.57$ per cent. In computing the value of H, we note that the least precise factor is the current which is uncertain by 0.48 per cent., and hence should be carried in the computation to four significant figures. H should therefore be computed by four place logarithms, four figures being retained in each factor, including the constant for transforming Joules to calories.

$$\begin{aligned} H &= 0.2390 \times 2.501 \times 109.7 \times 3600 \\ &= 236100 \text{ calories.} \end{aligned}$$

The numerical deviation Δ in H is obtained at once from its fractional deviation, as

$$\Delta = 240{,}000 \times 0.0057 = 1400 \text{ calories.}$$

Problem 3.—The candle power of a gas flame is measured against a standard candle by means of a photometer, the flame being placed at the end of a bar 100 inches from the candle. Suppose the mean of a series of disk settings gave $a = 20.17 \pm 0.27$ inches, a being the distance of the disk from the candle. Compute the candle power of the flame and its deviation, assuming that the candle is burning at its normal rate.

$$\frac{L \text{ (flame)}}{L_0 \text{ (candle)}} = \frac{(100-a)^2}{a^2}.$$

This may be written $L = L_0\left(\frac{100-a}{a}\right)^2 = \left(\frac{100-a}{a}\right)^2$

as $L_0 = 1$ candle power $= 1$ c.p. $=$ constant.

$$L = \left(\frac{100. - 20.17}{20.17}\right)^2 = \left(\frac{79.83}{20.17}\right)^2 = 15.67 \text{ c.p.}$$

L is a function of a single variable a; it is to be noted in the precision discussion that the function cannot be regarded as a fraction in which a deviation in the numerator is inde-

pendent of the deviation in the denominator. The formula may, however, be simplified by writing it in the form

$$L' = \sqrt{L} = 1 \times \frac{100 - a}{a}$$

and solving first for the deviation in L', after which the desired deviation in L can be easily found.

The deviation Δ' in L', due to a deviation $\delta = 0.27$ inch in a, is

$$\Delta' = \frac{d}{da}\left(1 \times \frac{100 - a}{a}\right) . \ \delta = 1 \times \frac{100}{a^2} . \ \delta$$

$$= 1(\text{c.p.})^{\frac{1}{2}} \frac{100 \text{ cm.}}{20^2 \text{ cm.}^2} \times 0.27 \text{ cm.} = 0.068 \ (\text{c.p.})^{\frac{1}{2}}.$$

To find now the deviation in L, we may proceed in either of two ways:—

First, General Method:

$$L' = \sqrt{L} = L^{\frac{1}{2}}$$

$$\therefore \Delta' = \frac{dL'}{dL} . \ \Delta = \frac{1}{2} L^{-\frac{1}{2}} . \ \Delta$$

where Δ is the desired deviation in L.

Therefore, $\qquad \Delta = 2\sqrt{L} . \ \Delta'$

$$= 2 \times \sqrt{16 \text{ c.p.}} \times 0.068 \ (\text{c.p.})^{\frac{1}{2}}$$

$$= 0.54 \text{ c.p.}$$

Second, Fractional Method:

The fractional deviation in L' corresponding to Δ' is

$$\frac{\Delta'}{L'} = \frac{0.068}{\sqrt{L}} = \frac{0.068}{\sqrt{16}} = 0.017;$$

and, since $L' = L^{\frac{1}{2}}$, the fractional deviation in L' is one-half the fractional deviation in L;

$$\frac{\Delta'}{L'} = \frac{1}{2} . \frac{\Delta}{L},$$

$$\therefore \frac{\Delta}{L} = 2 . \frac{\Delta'}{L'} = 2 \times 0.017 = 0.034.$$

Therefore, $\Delta = 0.034 \times L = 0.034 \times 16$ c.p. $= 0.54$ c.p.

The same result would of course be obtained by applying the general differentiation method to the original formula for L, but the resulting value of the differential coefficient is somewhat more complicated.

On the assumption that the average light emitted by the candle during the measurements is equal to one candle power, the candle power of the gas flame is 15.67 ± 0.54 c.p.; *i.e.*, it is known to only $100 \frac{0.54}{16} = 3.4$ per cent., although the original photometer setting a is good to $100 \frac{0.27}{20} = 1.4$ per cent.

Problem 4.—Suppose the index of refraction n of a substance is to be determined by measuring the angle of incidence i and the angle of refraction r of a ray of light. If approximate measurements give $i = 45°$ and $r = 30°$, how precisely should these two angles be measured to give n to 0.2 per cent.?

The formula for n is

$$n = \frac{\sin i}{\sin r}.$$

As this is not a product function of the variables i and r, we must use the general deviation method if we treat the formula in the above form. If, however, we change variables to x and y, letting $x = \sin i$ and $y = \sin r$, the formula becomes $n = \frac{x}{y}$, to which we may apply the fractional method of solution for finding the allowable deviations in x and y. Having done this, however, we still have two new problems to solve by the general method, namely, the determination of the deviations in i and r from the equations $x = \sin i$ and $y = \sin r$. Both methods lead, of course, to the same result. We will solve the problem both ways.

First Solution. General Method:

$$n = \frac{\sin i}{\sin r}.$$

Given $100 \frac{\Delta}{n} = 0.2$; $i = 45°$; $r = 30°$; to find δ_i and δ_r.

We must first find the value of the Δ in n from the prescribed percentage precision before proceeding to the solution. This necessitates knowing the approximate value of n, which is easily obtained from the data;

$$n = \frac{\sin 45}{\sin 30} = \frac{1}{\sqrt{2}} \div \frac{1}{2} = 1.4 \text{ approximately.}$$

Hence $\Delta = 1.4 \times 0.0020 = 0.0028$.

Distributing this deviation between i and r by equal effects,

$$\Delta_i = \Delta_r = \frac{\Delta}{\sqrt{2}} = \frac{0.0028}{1.4} = 0.0020.$$

But $$\Delta_i = \frac{\partial n}{\partial i} \cdot \delta_i = \frac{\cos i}{\sin r} \cdot \delta_i$$

$$\therefore \delta_i = \Delta_i \frac{\sin r}{\cos i} = 0.0020 \frac{\sin 30}{\cos 45} = 0.0014$$

To express this allowable deviation in the angle in *degrees*, we note that

$$1° = \frac{\pi}{180} = 0.017 \text{ radians},$$

therefore, $\delta_i = \dfrac{0.0014}{0.017} = 0.082°$, or $4.9'$.

Similarly, $\Delta_r = \dfrac{\partial n}{\partial r} \cdot \delta_r = \dfrac{\sin i \ \cos r}{\sin^2 r} \cdot \delta_r$

$$\therefore \delta_r = \Delta_r \cdot \frac{\sin^2 r}{\sin i \ \cos r} = 0.0020 \frac{\sin^2 30}{\sin 45 \ \cos 30} = 0.00082,$$

or, expressed in degrees, $\delta_r = \dfrac{0.00082}{0.017} = 0.048°$, or $2.9'$.

The solution shows, therefore, that, to obtain a precision of 0.2 per cent. in n, an instrument should be used capable of reading to at least $3'$. In practice one graduated to read to minutes would be chosen.

Second Solution. Fractional Method: If we put $x = \sin i$ and $y = \sin r$, then $n = \dfrac{x}{y}$, and we may use the inspection method as follows. The prescribed fractional deviation is stated to be not greater than $\dfrac{\Delta}{n} = 0.0020$. Hence, distributing this deviation between x and y by equal effects, we have

$$\frac{\Delta_x}{n} = \frac{\Delta_y}{n} = \frac{1}{\sqrt{2}} \cdot \frac{\Delta}{n} = \frac{1}{\sqrt{2}} \cdot 0.0020 = 0.0014.$$

But by inspection of $n = \dfrac{x}{y}$ it is seen that

$$\frac{\Delta_x}{n} = \frac{\delta_x}{x} \text{ and } \frac{\Delta_y}{n} = \frac{\delta_y}{y}.$$

Hence $$\frac{\delta_x}{x} = 0.0014 \text{ and } \frac{\delta_y}{y} = 0.0014.$$

We have now two new problems to solve, namely, to find δ_i and δ_r from the above values of the allowable precision in

x and y. As $x = \sin i$ is a trigonometric function, we must go back to the general differentiation method to find the deviation in i corresponding to a deviation δ_x in x.

As $$\frac{\delta_x}{x} = 0.0014,$$

we have $$\delta_x = 0.0014\, x = 0.0014 \sin 45° = 0.0010.$$

Also $$\delta_x = \frac{d \sin i}{di} \cdot \delta_i = \cos i \cdot \delta_i$$

$$\therefore \delta_i = \frac{\delta_x}{\cos i} = 0.0010 \div \frac{1}{\sqrt{2}} = 0.0014,$$

or in degrees $$\delta_i = \frac{0.0014}{0.017} = 0.082° = 4.9'.$$

Similarly, to find δ_r, given $y = \sin r$ and $\frac{\delta_y}{y} = 0.0014$, we have

$$\delta_y = 0.0014\, y = 0.0014 \sin 30° = 0.00070.$$

Also $$\delta_y = \frac{d \sin r}{dr} \cdot \delta_r = \cos r \cdot \delta_r$$

$$\therefore \delta_r = \frac{\delta_y}{\cos r} = 0.00070 \div \frac{\sqrt{3}}{2} = 0.00082,$$

or in degrees $$\delta_r = \frac{0.00082}{0.017} = 0.048° = 2.9'.$$

In this problem there is evidently no saving of labor by transforming the function to the product form and first using the fractional method, as the ultimate solution necessitates going back to the general method.

PROBLEMS

Questions and Problems.

1. Explain the terms: *precision measure; deviation measure; constant error; residual error; probable error; mean error; huge error; indeterminate error; weighted mean; weights.*

2. What is the geometrical significance of the average deviation, probable error, and mean error in relation to the curve representing the law of chance?

3. Is it practicable to reduce the average deviation of a mean result to any desired value by increasing the number of observations? Why?

4. If the mean value of the length of a rod computed from nine measurements is 24.213 cm. $A.D. = 0.012$ cm., how many more similar observations should be made in order that the A.D. $= 0.0060$ cm.?

5. Under what circumstances may an observation properly be rejected, and why?

6. What determines the number of places of significant figures to be retained at any part of a computation?

Under what circumstances should four, five, or seven-place logarithms be used in a computation?

7. Do the number of significant figures in a result depend upon the position of the decimal point? Explain reasons for your answer. Does the *precision* of a result depend upon the position of the decimal point? Why?

8. Explain why when adding or subtracting observed quantities we are governed by *decimal* places in rejecting figures, but, when multiplying or dividing, places of significant figures, regardless of the decimal point, must be considered.

9. Given the following measurements and their deviation measures:—

241.631 gms.	reliable to 0.25 per cent.
1620.124 gms.	$A.D. = 0.81$ gms.
10.005 gms.	reliable to 1 part in 1,000
7141.110 gms.	$P.E. = 0.603$ gms.

(*a*) Indicate all superfluous figures.

(*b*) Which is the most and which the least precise quantity, and why?

(*c*) How many figures should be retained in each quantity in computing their sum?

(*d*) How many figures should be retained in each quantity in computing their product?

(*e*) How many figures should be retained in each quantity in computing the difference between the product of first two and the product of the last two quantities?

10. Given the following measurements of the length of a rod and their precision measures:—

24.316 cm.	$A.D. = 0.028$ cm.
24.3922 cm.	fractional precision 12 parts in 1,000
24.358 cm.	$P.E. = 0.0121$ cm.
24.3091 cm.	reliable to 0.11%.
24.3100 cm.	$A.D. = 0.0172$ cm.

(*a*) Indicate any superfluous figures.

(*b*) Indicate the order of reliability of the results.

(*c*) Compute the relative "weights."

(*d*) Compute the weighted mean.

11. The precision measures of four independent determinations of the modulus of elasticity of steel are expressed as follows: 1st, 0.60 per cent.; 2d, 2 parts in 1,000; 3d, probable error $= 4.2\,\frac{Kgm.}{mm.^2}$; 4th, average deviation $= 10.\,\frac{Kgm.}{mm.^2}$. The modulus of elasticity is about $20{,}000\,\frac{Kgm.}{mm.^2}$.

Which measurement is the most reliable? Find the relative weights of the four determinations.

12. Independent determinations of the rate of a pendulum by different methods and observers gave

0.70061 sec.	$A.D.$	$= 0.00023$ sec.
0.70047 "	$A.D.$	$= 0.00069$ "
0.70056 "	correct to 0.092%	
0.70051 "	$P.E.$	$= 0.00039$ sec.

(*a*) Which is the most reliable observation?

(*b*) Compute the relative "weights" of the observations.

(*c*) Indicate how to compute the best representative value of all the observations.

13. The dimensions of a right cylinder are found to be as follows:—

length	$= 12.183$ cm.	$A.D. = 0.024$ cm.
diameter	$= 4.242$ cm.	$A.D. = 0.021$ cm.

Find the volume and its deviation measure, indicating proper number of significant figures at each step of the computation.

14. The diameter of a spherical globe is found to be approximately six inches. If the average diameter varies by 0.1 per cent., what variation in cubic inches will this produce in the volume? If the variation in the diameter is 0.0020 inch, what uncertainty will result in the volume?

15. The length of a physical pendulum is given by the expression

$$l = h + r + \tfrac{2}{5}\cdot\frac{r^2}{h + r}$$

where $r = \frac{1}{2}$ diameter of ball $= \dfrac{d}{2}$,

and h = distance of knife edge from the top of the ball.

Suppose	$h = 100.031$ cm.	$A.D. = 0.027$ cm.
	$d = 6.256$ cm.	$A.D. = 0.022$ cm.

(*a*) Find the resultant deviation in l.

(*b*) How many significant figures should be retained in each term of the formula, in computing l, and why?

(*c*) Is it necessary to consider the third term in this formula in a precision discussion of l? Why?

16. The mean time of nine coincidences of two beating pendulums is 50.43 seconds. $A.D. = 0.11$ second. The standard pendulum beating true seconds gains on the other pendulum. Compute the true time of vibration of the latter and its precision measure.

17. The time of swing of a half-second pendulum is measured to 0.20 per cent. The length is measured to 0.10 mm. Find the precision of the computed value of g.

18. Measurements with a spherometer gave for a certain lens

$$h = 1.22110 \text{ mm.} \quad A.D. = 0.00088 \text{ mm.}$$
$$r = 35.735 \text{ mm.} \quad A.D. = 0.061 \text{ mm.}$$
$$R = \frac{r^2}{2h} + \frac{h}{2}.$$

Compute R, the radius of curvature of the lens retaining the proper number of significant figures throughout computation. What deviation would be introduced in R by a deviation of 0.00088 mm. in h? By a deviation of 0.061 mm. in r? What would be the combined effect of these deviations on R? Is the term $\frac{h}{2}$ negligible in the above precision discussion, and if so, why?

19. A 10 ohm coil is standard at 15° C. What will be its resistance at 30° C. and how precise will this be known if the temperature is determined to 0.1 degree?

$$R_t = R_{15}\left[1 + 0.00388\,(t - 15)\right].$$

How closely must t be known in order that the resistance may be depended upon to 0.02 per cent.?

20. The electromotive force of a Clark cell is

$$E_t = 1.4340\left[1 - 0.00078\,(t - 15)\right].$$

How closely must t be measured in order that E may be known to 0.05 per cent.?

21. Given a coil of wire the resistance of which at 15° C. may be assumed to be exactly 100 ohms. The wire is an alloy the resistance of which at $t°$ may be computed by the expression

$$R_t = R_{15}\left[1 + 0.00051\ (t - 15)\right].$$

Calculate the amount of heat H generated in the wire in an hour's run by a current of 11.273 amperes, if the wire is maintained at a temperature of 45° C. $H = I^2Rt$.

If the temperature is known to ±1.0°, the duration of the run to ±1.0 second, and the current to ±0.011 ampere, calculate the deviation in H in Joules and in per cent. Are the deviations in any measurements negligible, and if so, why?

22. The sensitiveness of a spirit level is $\theta = \frac{h}{l} \times 206265''$.

(*a*) If $l = 380.$ mm. approx. and $h = 0.0597$ mm. ± 0.0018 mm., how many figures would you keep in the value θ?

(*b*) How many would you use in the value of the radian, 206265? How many in h?

(*c*) If l and 206265 are to introduce *no* deviation in the result in comparison with h, what is the allowable percentage deviation, and how few figures may be used in each?

(*d*) If you wished to determine θ with greater precision than above, in which measurement would you use greater care?

23. In calibrating a burette by drawing off water for each 10 cc. and weighing it in a flask, how precise should the weighings be made if the calibration is to be reliable to 0.01 cc.? Suppose the calibration were made with mercury, how close should the weighings be made? Is it necessary to note the temperature of the water in this calibration and why?

24. It is desired to calibrate a flask to hold 1,000 cc. at 20° C., the calibration to be reliable to 0.5 cc. What weight (using brass weights sp. gr. = 8.5) would you add to the weight of the flask empty, so that it would exactly balance the water having the desired volume? How precise would you make your weighings? Would it be necessary to take

into account the barometer reading in figuring the correction for reduction to vacuo? Why?

25. The per cent. of silver in a certain alloy is determined gravimetrically by weighing the amount of silver present as silver chloride. Suppose an analysis gave the following results:—

Wt. of alloy		1.43252 gms.	± 0.00014 gm.
"	" AgCl	0.19513 gms.	± 0.00021 gms.

Compute the per cent. of silver in the alloy and the precision with which this would be known.

Atomic weight silver $= 107.93 \pm 0.02$
Atomic weight chlorine $= 35.45 \pm 0.03$

26. A specific gravity determination by Archimedes' principle gave the following results:—

weight of substance in air = 10.2431 gms. $\pm$ .0004 gm.
weight of substance in distilled water at 20° C. = 9.0422 gms. $\pm$ .0010 gm.
density of water at 20° C. = 0.99825.

Compute specific gravity and its precision measure, using the correct number of places of significant figures throughout the computation. Is the correction for reduction of weighings to vacuo negligible in this case, and why?

27. Given

$$w_t = (w_{20} - b)\left[1 + k\,(t^\circ - 20^\circ)\right]\frac{D_t}{D_{20}}$$

where $w_{20} = 27.6231$ grams = weight of bottle plus water.
$b = 12.6193$ grams = weight of bottle.
$k = 0.000026$ per degree C.
D_t and D_{20} = density of water at t° and 20° respectively.

The density D diminishes 0.02 per cent. per degree C. What is the greatest allowable value which t may have and the correction term due to expansion of glass be negligible; *i.e.*, affecting the value of $(w_{20} - b)$ by less than 0.0001 gm.? Is the correction due to change of density negligible for $t = 22^\circ$

and why? If $D_{20} = 0.99827$ and $t = 25°$, compute w_t, *using as few figures as practicable.*

28. With what precision would it be necessary to measure the length and diameter of a right cylindrical column approximately 12 inches long and 6 inches in diameter in order to determine the volume to 0.10 per cent.? What deviation in cubic inches would this correspond to?

29. If it is desired to compute the area of a circle approximately 10 sq. cm. in area to 5 parts in 10,000, how precise should the diameter be known? How many places should be retained in π in the computation?

30. If the volume of a sphere is computed from a measurement of its diameter, how precise should the latter be measured in order that the former may be reliable to 0.1 per cent.; to 1 part in 500?

If the sphere is approximately six inches in diameter, what precision in the diameter does a precision of 0.1 per cent. in the volume require? If the *A.D.* of the diameter is 0.022 inch, what is the precision of the volume?

31. The ratio of the length of the arms of a balance is given by the expression

$$r = \frac{\text{length right arm}}{\text{length left arm}} = \frac{\sqrt{W_l}}{\sqrt{W_r}},$$

where W_l and W_r are the observed weights of a given mass when weighed in the left and right hand pan, respectively. If the mass weighs approximately 20 grams, with what precision must W_l and W_r be determined in order that the ratio r may be reliable to 0.01 per cent.?

32. The specific gravity of a platinum alloy is desired to 0.1 per cent. The method based on Archimedes' principle is to be used. The weight of substance in air is about 42 grams. The specific gravity is approximately 21. To what fraction of a gram should each weighing be made? What corrections should be applied?

33. A freely falling body passes through the distance $h = \frac{1}{2}gt^2$ in t seconds. It is desired to determine g to one-tenth per cent. from a measurement of h and t.

(*a*) How precisely should h and t be measured?

(*b*) If $h = 4$ meters, what will be the allowable numerical deviation in h and t?

(*c*) If $h = 4$ meters and it is found that $\delta_h = 1.0$ mm. and $\delta_t = 0.0014$ second, what will be the resultant deviation in g?

34. If g is to be computed from the mean of a series of nine observations on the time of swing of a pendulum, the length of which is $l = 1$ meter, what must be the percentage precision of t and l in order that g be precise to 0.1 per cent.? What will be the value of *ad.* and *A.D.* in the case of t? How many figures should be retained in π?

35. The gas constant R is given by the formula

$$R = \frac{pv}{T} \text{ where } T = t^\circ + 273^\circ.$$

(*a*) How precisely must p, v, and t be measured in order that R may be reliable to 0.1 per cent.?

(*b*) How large a deviation will be introduced in R by a variation of 1° C. in the temperature alone at 20° C.?

36. The indicated horse power (I. H. P.) of an engine is given by the expression $I.\ H.\ P. = \dfrac{P \times L \times A \times N}{33{,}000}$.

Suppose for a given engine the approximate values of these quantities are as follows:—

$P = 50$ lbs. per sq. in. = mean effective pressure.
$L = 2$ feet = length of stroke.
A = Area of piston, the diameter, D, of which is 16 inches.
$N = 100$ = number of strokes per minute.

How, precisely, should P, L, D, and N be determined in order that the computed horse power of the engine may be reliable to 1 per cent.? To one-quarter of a horse power?

37. A rectangular steel rod of breadth b and depth d is supported at its ends and loaded at its centre by a weight W.

If l is the length of the rod between its supports and a is the deflection at the centre,

$$a = \frac{Wl^3}{4Ebd^3},$$

where E is the modulus of elasticity.

Suppose measurements of these quantities gave

$b = 8.113$ mm.	$\delta_b = 0.042$ mm.
$d = 10.50$ mm.	$\delta_d = 0.025$ mm.
$l = 1.000$ meter	precise to one part in 5,000
$a = 2.622$ mm.	precise to 0.25%
$W = 2$ kgms.	precise to 0.02 gram

Compute

(*a*) E, the modulus of elasticity.
(*b*) The deviation in E due to each component.
(*c*) The resultant deviation in E.

38. What would be the allowable deviations in the measurement of b, d, l, and a for the beam defined in problem 37 if the value of E is to be reliable to $\frac{1}{2}$ per cent.? Assume the error in the value of W to be negligible.

Do you think this precision could be readily attained? Why?

39. The modulus of elasticity E of a cylindrical wire, of length l, cross section q, which when loaded with a weight w is elongated by an amount a, is given by the expression:

$$E = \frac{lw}{aq}.$$

$a = m_1 - m_0$ where m_1 and m_0 are mean micrometer readings when the wire is under a load of w kilograms and no load, respectively. $q = \frac{1}{4}\pi d^2$ where d is the mean diameter of the wire.

Given the following data:

$l = 200.11$ cm.	$a.d. = 0.05$ cm.
$w = 10$ kilograms	accurate to 1 gram.
$m_1 = 9.4255$ mm.	$A.D. = 0.0024$ mm.
$m_0 = 8.2233$ mm.	$A.D. = 0.0012$ mm.
$d = 1.002$ mm.	correct to 0.2%

(*a*) Compute the average deviation and the percentage deviation of q.

(*b*) Compute the deviation of a.

(*c*) Compute E, using proper number of significant figures.

(*d*) Compute resultant percentage error of E.

(*e*) Are any of the above data more precise than necessary and why?

40. (*a*) If it is desired to determine E for the above sample of wire to 0.5 per cent., how precisely should the components a and q be measured, assuming that the deviations in l and w can be made negligible?

(*b*) How precisely must d be measured to fulfil this condition?

(*c*) How precisely must m_1 and m_0 be measured?

(*d*) Do you think this degree of precision could be readily attained, and why?

41. It is desired to determine the focal length of a plano-convex lens from spherometer measurements. $n = 1.5$ approximately.

$$\frac{1}{F} = (n-1)\left(\frac{1}{R} + \frac{1}{R'}\right);$$

$$R = \frac{r^2}{2h} + \frac{h}{2}.$$

(*a*) If F is desired to 0.1 per cent., how precisely must n and R be determined?

(*b*) If preliminary measurements give $r = 40$ mm. and $h = 4$ mm., approximately, how precisely should r and h be determined to fulfil condition *a*?

42 The formula for computing the wave length by means of a diffraction grating is for second order spectra $\lambda = \frac{1}{2n} \cdot \sin \theta$. If the grating is ruled 17,296 lines to the inch, and a preliminary measurement gives $\theta = 53°$ approximately for the sodium band, with what precision must θ be measured, and how should the optical circle be graduated to give λ to one part in 10,000?

43. $I = K \tan \theta$ for a tangent galvanometer, whose galvanometer constant is

$$K = 1.963 \pm 0.002.$$

If the deviation in reading any deflection θ is $\delta_\theta = 0.1°$, compute the value of I and its deviation measure for $\theta = 45°$ and $\theta = 60°$.

44. The heating effect H of an electric current is to be calculated from measurements of I, t, and E, or I, t, and R.

$$H = 0.2390\, IEt\,;\ H = 0.2390\, I^2Rt.$$

(*a*) If H is to be reliable to 0.2 per cent., what must be the percentage precision of each of the component measurements in both cases? The allowable deviation in each measurement?

(*b*) If $E = 110$ volts and is measured to 0.5 per cent., $R = 100$ ohms and is reliable to 1 part in 1,000, $\delta_1 = 0.03$ amp. and $\delta_t = 0.2$ sec.; compute the precision of a ten-minute run by each method, and state which method you consider the better.

45 Given the following approximate values for a specific heat determination:—

s = sp. ht. of substance
w = wt. of substance = 300 grams
w_0 = " " water = 500 grams
w_1 = " " calorimeter = 100 grams
s_1 = sp. ht. of calorimeter = 0.095
$(t_s - t_2)$ = fall of temperature of substance
= $100° - 20° = 80°$ C.
$(t_2 - t_1$ = rise of temperature of calorimeter
plus water = $20° - 15° = 5°$ C.

$$s = \frac{(w_0 + k)\ (t_2 - t_1)}{w\ (t_s - t_2)},\ \text{where } k = w^1 s_1$$

If s is desired to 0.5 per cent., with what precision should each of the quantities w, w_0, w_1, s_1, t_s, t_1, and t_2, be measured or known?

Can the deviations in any of the above quantities be readily made negligible? Solve the problem under these conditions.

46. The resistance of a metal bar is determined by measuring the drop in potential between its ends and the corresponding current flowing through it. Suppose mean measurements gave

$$I = 11.431 \text{ amperes} \pm 0.022 \text{ ampere},$$
$$E = \ 0.5073 \text{ volt} \pm 0.010 \text{ volt}.$$

Compute the resistance of the bar and its deviation in ohms.

47. Suppose the mean of nine settings of a photometer disk gave

$$a = 240.1 \text{ cm.} \pm 1.1 \text{ cm.},$$

a being the distance of the disk from a gas flame whose candle power, L, is desired, and $300-a$, the distance of the disk from a standard candle. Assuming the candle to be burning at its normal rate, compute the candle power of the flame and its percentage deviation.

$$\frac{L \text{ (flame)}}{L' \text{ (candle)}} = \frac{a^2}{(300-a)^2}.$$

48. If $E = K \log_e \frac{p_1}{p_2}$, compute the deviation in E due to a deviation of 0.1 per cent. in the measurement of p_1 and p_2 respectively, K being a constant. If the deviations in p_1 and p_2 were known to be of the same sign, what would be their resultant effect on E?

49. Suppose a stop watch loses regularly at the rate of 1.2 seconds in 15 minutes and the uncertainty of stopping and starting the second hand is ± 0.1 second in each case. If a runner makes a one-mile record in 4 minutes $35\frac{4}{5}$ seconds by the watch, calculate the true time and its deviation measure in seconds and per cent.

50. To what fraction of a gram should a piece of aluminum be weighed in air and in water, respectively, in order that its computed specific gravity may be reliable to one part in a thousand? The approximate weight of the sample in air is 25 grams, and its specific gravity is approximately 2.7.

51. Four independent observers determine a resistance of about one ohm to 0.10 per cent., 0.0030 ohm, one part in five

hundred, and with a probable error of 0.00085 ohm, respectively. What are the relative reliabilities of the determinations and their respective weights?

52. A certain 32 c.p. lamp takes a current of approximately 1 ampere at 110 volts. If its resistance under these conditions is desired to 0.5 ohm, how precise should the current and voltage be measured?

53. The mean of sixteen comparisons of a yard scale and a standard meter scale gave the result:

$$1 \text{ yard} = 0.91449 \text{ m.}; \quad A.D. = 0.00011 \text{ m.}$$

If there is a residual error of ± 0.007 cm. in the meter scale after all sources of constant error have been corrected for, to what fraction of an inch is the value of the yard reliable? If the meter scale were correct at 0° C. and used at 20° C., how large a constant error (in inches) would result if its expansion were neglected, assuming that it expanded 0.0025 per cent. per degree?

54. Two tuning forks of slightly different pitch "beat" with each other 45.31 ± 0.45 times per minute when sounded simultaneously. If the standard fork makes exactly 256 vibrations per second, what is the rate of the second fork and its deviation measure, assuming it to be of lower pitch than the standard?

55. Given the following data on the specific gravity of a substance lighter than water.

Weight of substance in air	$= 10.1321$ gms. ± 0.0002 gm.
Weight of substance and sinker immersed in water at 20.0° C.	$= 8.4418$ gms. ± 0.0020 gm.
Weight of sinker immersed in water at 20.0° C.	$= 10.4522$ gms. ± 0.0010 gm.

Compute the specific gravity of the substance referred to water at 20.0° C. and its numerical and percentage deviation. If the density of water decreases 0.18% between 4° C. and 20° C., compute the specific gravity referred to water at 4° C. How precisely should the temperature of the water at 20° be determined if this last reduction is to be reliable to 0.05%?

56. Suppose a pendulum approximately 550 feet in length is swung from the top of the Washington Monument. How precisely should the length and time of vibration be measured in inches and seconds, respectively, if the value of g computed from these data is to be reliable to $0.50 \dfrac{\text{inch}}{(\text{second})^2}$?

57. Suppose that fifty 16 c.p. incandescent lamps are used on the average 2 hours a day for 4 weeks. Each lamp takes 0.5 ampere at 110 volts. Calculate the total amount of energy consumed in Joules. If this energy is measured by determining the average current and voltage by an ammeter and voltmeter each of which reads uniformly 2% too high, how much overcharge would there be on the lighting bill if the cost of electrical energy is 10 cents per kilowatt hour?

If the average voltage and current are uncertain by ± 1.1 volts and ± 0.0050 ampere, respectively, what uncertainty would there be in the total electrical energy consumed, and in its value in dollars and cents?

58. If the weight of a substance in air is

$$w = 49.7631 \pm 0.0012 \text{ grams}$$

and it is desired to calculate its weight W in vacuo, how closely should the density δ of the substance, the density Δ of the balance weights, and the density σ of the air in the balance case be known in order that the *correction* to be added to w may be computed to the nearest 0.0005 gram?

Given $\delta = 0.8$, $\Delta = 8.4$, and $\sigma = 0.0012$ approximately.

$$W = w\left[1 + \sigma\left(\frac{1}{\delta} - \frac{1}{\Delta}\right)\right]$$

59. The formula for calculating the temperature t of a mercurial thermometer when corrected for stem exposure is

$$t = t_1 + 0.000156\,(t_1 - t_a)n,$$

where t_1 = observed temperature,
t_a = temperature of exposed stem,
n = number of degrees exposed at $t_a°$.

Suppose $t_1 = 330° \pm 0.5°$, $t_a = 30° \pm 0.5°$, and $n = 200° \pm 5°$ approximately.

Compute t and its deviation measure.

60. The formula for the latent heat of vaporization is

$$r=\frac{(w_0+k)\ (t_2-t_1)-w\,(t_s-t_2)}{w}$$

If the rise in temperature $t_2-t_1=25°-15°=10°$, the fall in temperature of steam $t_s-t_2=100°-25°=75°$, the condensed steam $=20$ grams, and the water equivalent $w_0+k=1{,}200$ grams approximately, calculate how precisely you would determine each of these four factors if r is to be reliable to 0.5%. (Assume $r=$ 540 cal. for steam.)

61. If $I=\dfrac{E}{R_1+R_2+R_3}$, what is the allowable deviation in ohms in R_1, R_2, and R_3, respectively, if I is to be reliable to 0.1% and the deviation in E is negligible? The approximate values of the resistances are $R_1=10$ ohms, and $R_2=100$ ohms, and $R_3=1$ ohm. If the above formula were

$$I=\frac{E}{R_1\times R_2\times R_3},$$

what would be the solution under the same conditions?

62. Suppose $I_1=K_1 \sin\theta$ and $I_2=K_2\tan\theta$. The value of K_1 and K_2 are each known to 0.1%; $\theta=45°$; $\delta_\theta=\pm 0.1°$. Compare the precision of I as calculated by the two formulæ.

63. Suppose you had three areas, each equal approximately to 10 sq. in., one being a circle, one a square, and one an equilateral triangle. How precisely would you have to know the diameter of the first and the average value of one side of the last two, in order that the computed area may be reliable to 0.01 sq. in.?

64. The capacity of a spherical condenser is given by the expression $C=\dfrac{K r_1 r_2}{r_2-r_1}$. Suppose

$$r_1=10.0010 \text{ cms.} \pm 0.0019 \text{ cm.};$$
$$r_2=15.0000 \text{ cms.} \pm 0.0044 \text{ cm.};$$
$$K=\ 2.0130\pm 0.0012.$$

Calculate C and its resultant deviation measure; the percentage deviation in the numerator; the deviation measure of the denominator expressed in centimeters.

65. Given a triangular lot of land whose three sides are a, b, c, respectively. $a = 120$ feet approximately; $b = 180$ feet approximately. The angle θ between a and b is about 45°. It is desired to find the area of the triangle to 0.12%. How precisely should a, b, and θ be measured? If a, b, and θ are measured with the above precision, what would be the precision of the computed value of c?

$$\text{Area} = \tfrac{1}{2}ab \sin \theta$$
$$c^2 = a^2 + b^2 - 2ab \cos \theta$$

66. A sample of sodium chloride, NaCl, is analyzed by precipitating with silver nitrate and weighing the silver chloride, AgCl.

Wt. of sample of NaCl = 0.5017 gm. ± 0.0005 gm.
Wt. of AgCl = 1.1817 gms. ± 0.0012 gm.

Assuming the atomic weight of sodium and chlorine to be known to ±0.1 per cent. and that of silver to be known to ±0.03 per cent., calculate the per cent. of chlorine present in the sample and the precision with which you can depend upon this result.

Atomic weights:
Na = 23.00 Cl = 35.46 Ag = 107.88

67. The index of refraction of a substance is given by the expression $n = \dfrac{\sin i}{\sin r}$.

If $i = 45° \pm 10'$ and $r = 30° \pm 5'$, compute the value of n and its average and percentage deviation.

68. The index of refraction of a substance as determined by the Pulfrich refractometer is given by the relation

$$n = \sqrt{N^2 - \sin^2 \theta},$$

where N is a constant of the instrument and θ is the measured angle.

If $N = 1.62100 \pm 0.00005$ and $\theta = 45°$, approximately, how precisely should θ be measured in order that n may be reliable to 0.1 per cent.?

69. Given two resistances A and B, each known to $\frac{1}{20}$ per cent; $A = 2$ ohms approximately, $B = 1$ ohm approximately.

If settings on a slide wire bridge are $l_A = 660$ mm., and $l_B = 340$ mm. approximately and these are each known to ± 0.2 mm., compute the deviation measure of n.

$$n = \frac{l_A B - l_B A}{A - B}$$

70. What considerations determine the precision with which the constants in the equation of a straight line may be obtained by the graphical method? With plotting-paper 10 inches on a side and ruled in twentieths of an inch, explain what the extreme limits of precision attainable are.

71. What is a residual plot, and what is its use? Explain fully.

72. How would you determine by a graphical method the value of the constants a, b, and c in the equation

$$y = a + bx + cx^2$$

from a series of values, x_1, y_1, x_2, y_2, etc.?

73. What is a logarithmic plot and to what class of problems is it applicable? Explain fully its use by an illustration, first, using rectangular co-ordinate paper, and second, using logarithmic paper.

74. The heat H generated in a coil of wire in a given time varies as the square of the current I. How would you test this relation graphically by a series of determinations of H and I?

75. How would you test graphically the law that the deflection of a beam loaded at its centre and supported at its ends is proportional to the cube of its length?

76. From a series of measurements on the time of vibration and corresponding length of a pendulum explain how you could obtain the mean value of g by treating the data graphically. $t = \pi \sqrt{\frac{l}{g}}$.

77. From a series of measurements on the strength of current I flowing in a circuit of resistance R to which a variable

(measured) electrometer force E is applied, explain how you would find the mean value of R by treating the data by a graphical method.

78. Suppose a current I is measured by a tangent galvanometer for which $I=K \tan \theta$. From a series of values of I and corresponding values of θ explain how you would find K by a graphical method.

79. The formula for the focal length of a lens is

$$\frac{1}{f}=\frac{1}{p}+\frac{1}{p'}.$$

From a series of measurements on p and p' explain how you would find the mean value of f by means of a graphical method.

APPENDIX.

TABLE I.

MATHEMATICAL CONSTANTS.

Quantity.	Value.	Remark.	Common Logarithm.
ϵ	2.71828	Base of Napierian, natural or hyperbolic logarithms.	0.434295
$\log_{10} \epsilon$	0.434295	Factor to multiply Napierian logs to convert into Common logs.	9.637784
$\dfrac{1}{\log_{10} \epsilon}$	2.30259	Factor to multiply Common logs to convert into Napierian logs.	0.362216
π	3.14159	Ratio of circumference to diameter.	0.497150
$1/\pi$	0.318310	Reciprocal of π.	9.502850
π^2	9.86960	Square of π.	0.994300
$\sqrt{\pi}$	1.77245	Square root of π.	0.248575
1 radian	$57°17'45''$	$57°.2958 = 206265'' =$ arc equal to radius.	
1 degree	$\dfrac{\pi}{180} = 0.017,4533$ radian		

TABLE II.

APPROXIMATION FORMULÆ.

It frequently happens in a computation that a factor of the general form $(1 \pm a)^n$ enters where n is a constant and a is a quantity whose numerical value is small compared with unity. In such cases the approximate value of the factor given by the first two terms of its expansion may usually be substituted in place of the factor itself without introducing an appreciable error in the result, and the computation becomes thereby decidedly simplified. If the factor is of the form $(m \pm a')^n$ where a' is small compared with m, it may be written

$m^n \left(1 \pm \frac{a'}{m}\right)^n = m^n \ (1 \pm a)^n$ and so reduced to the first form.

Table II. contains the approximate forms of the factor for several values of n, together with the error which would be introduced by using the approximation.

Factor.	Approximate Form.	Resulting Error.	Computed Error if a = 0.01.
$(1 \pm a)^n$	$1 \pm a$	$\frac{n\,(n-1)}{2}\,a^2$	
$(1 \pm a)^2$	$1 \pm 2a$	a^2	0.0001
$(1 \pm a)^3$	$1 \pm 3a$	$3a^2$	0.0003
$(1 \pm a)^4$	$1 \pm 4a$	$6a^2$	0.0006
$(1 \pm a)^{\frac{1}{2}}$	$1 \pm \frac{1}{2}a$	$-\frac{1}{8}a^2$	−0.00001
$(1 \pm a)^{\frac{1}{3}}$	$1 \pm \frac{1}{3}a$	$-\frac{1}{9}a^2$	−0.00001
$(1 \pm a)^{-1}$	$1 \mp a$	a^2	0.0001
$(1 \pm a)^{-2}$	$1 \mp 2a$	$3a^2$	0.0003
$(1 \pm a)^{-\frac{1}{2}}$	$1 \mp \frac{1}{2}a$	$\frac{3}{8}a^2$	0.00004
$(1 \pm a)^{-\frac{1}{3}}$	$1 \mp \frac{1}{3}a$	$\frac{2}{9}a^2$	0.00002

For $(1 \pm a)\ (1 \pm b)\ (1 \pm c)$ use $(1 \pm a \pm b \pm c)$.

For $\frac{(1 \pm a)\ (1 \pm b)}{(1 \pm c)\ (1 \pm d)}$ . . . use $(1 \pm a \pm b \mp c \mp d \ldots)$

For $\sqrt{m_1\, m_2}$ use $\frac{m_1 + m_2}{2}$ when m_1 and m_2 are nearly equal.

TABLE III.

SQUARES, CUBES, RECIPROCALS.

No.	Square.	Cube.	Recip.	No.	Square.	Cube.	Recip.
1.0	1.00	1.00	1.00	5.5	30.3	166.	.182
1.1	1.21	1.33	0.909	5.6	31.4	176.	.179
1.2	1.44	1.73	.833	5.7	32.5	185.	.175
1.3	1.69	2.20	.769	5.8	33.6	195.	.172
1.4	1.96	2.74	.714	5.9	34.8	205.	.169
1.5	2.25	3.38	.667	6.0	36.0	216.	.167
1.6	2.56	4.10	.625	6.1	37.2	227.	.164
1.7	2.89	4.91	.588	6.2	38.4	238.	.161
1.8	3.24	5.83	.556	6.3	39.7	250.	.159
1.9	3.61	6.86	.526	6.4	41.0	262.	.156
2.0	4.00	8.00	.500	6.5	42.3	275.	.154
2.1	4.41	9.26	.476	6.6	43.6	287.	.152
2.2	4.84	10.6	.455	6.7	44.9	301.	.149
2.3	5.29	12.2	.435	6.8	46.2	314.	.147
2.4	5.76	13.8	.417	6.9	47.6	329.	.145
2.5	6.25	15.6	.400	7.0	49.0	343.	.143
2.6	6.76	17.6	.385	7.1	50.4	358.	.141
2.7	7.29	19.7	.370	7.2	51.8	373.	.139
2.8	7.84	22.0	.357	7.3	53.3	389.	.137
2.9	8.41	24.4	.345	7.4	54.8	405.	.135
3.0	9.00	27.0	.333	7.5	56.3	422.	.133
3.1	9.61	29.8	.323	7.6	57.8	439.	.132
3.2	10.2	32.8	.313	7.7	59.3	457.	.130
3.3	10.9	35.9	.303	7.8	60.8	475.	.128
3.4	11.6	39.3	.294	7.9	62.4	493.	.127
3.5	12.3	42.9	.286	8.0	64.0	512.	.125
3.6	13.0	46.7	.278	8.1	65.6	531.	.123
3.7	13.7	50.7	.270	8.2	67.2	551.	.122
3.8	14.4	54.9	.263	8.3	68.9	572.	.120
3.9	15.2	59.3	.256	8.4	70.6	593.	.119
4.0	16.0	64.0	.250	8.5	72.3	614.	.118
4.1	16.8	68.9	.244	8.6	74.0	636.	.116
4.2	17.6	74.1	.238	8.7	75.7	659.	.115
4.3	18.5	79.5	.233	8.8	77.4	681.	.114
4.4	19.4	85.2	.227	8.9	79.2	705.	.112
4.5	20.3	91.1	.222	9.0	81.0	729.	.111
4.6	21.2	97.3	.217	9.1	82.8	754.	.110
4.7	22.1	104.	.213	9.2	84.6	779.	.109
4.8	23.0	111.	.208	9.3	86.5	804.	.108
4.9	24.0	118.	.204	9.4	88.4	831.	.106
5.0	25.0	125.	.200	9.5	90.3	857.	.105
5.1	26.0	133.	.196	9.6	92.2	885.	.104
5.2	27.0	141.	.192	9.7	94.1	913.	.103
5.3	28.1	149.	.189	9.8	96.0	941.	.102
5.4	29.2	157.	.185	9.9	98.0	970.	.101

TABLE IV.

FOUR PLACE LOGARITHMS.

Natural numbers.	0	1	2	3	4	5	6	7	8	9	Proportional Parts.								
											1	2	3	4	5	6	7	8	9
10	0000	0043	0086	0128	0170	0212	0253	0294	0334	0374	4	8	12	17	21	25	29	33	37
11	0414	0453	0492	0531	0569	0607	0645	0682	0719	0755	4	8	11	15	19	23	26	30	34
12	0792	0828	0864	0899	0934	0969	1004	1038	1072	1106	3	7	10	14	17	21	24	28	31
13	1139	1173	1206	1239	1271	1303	1335	1367	1399	1430	3	6	10	13	16	19	23	26	29
14	1461	1492	1523	1553	1584	1614	1644	1673	1703	1732	3	6	9	12	15	18	21	24	27
15	1761	1790	1818	1847	1875	1903	1931	1959	1987	2014	3	6	8	11	14	17	20	22	25
16	2041	2068	2095	2122	2148	2175	2201	2227	2253	2279	3	5	8	11	13	16	18	21	24
17	2304	2330	2355	2380	2405	2430	2455	2480	2504	2529	2	5	7	10	12	15	17	20	22
18	2553	2577	2601	2625	2648	2672	2695	2718	2742	2765	2	5	7	9	12	14	16	19	21
19	2788	2810	2833	2856	2878	2900	2923	2945	2967	2989	2	4	7	9	11	13	16	18	20
20	3010	3032	3054	3075	3096	3118	3139	3160	3181	3201	2	4	6	8	11	13	15	17	19
21	3222	3243	3263	3284	3304	3324	3345	3365	3385	3404	2	4	6	8	10	12	14	16	18
22	3424	3444	3464	3483	3502	3522	3541	3560	3579	3598	2	4	6	8	10	12	14	15	17
23	3617	3636	3655	3674	3692	3711	3729	3747	3766	3784	2	4	6	7	9	11	13	15	17
24	3802	3820	3838	3856	3874	3892	3909	3927	3945	3962	2	4	5	7	9	11	12	14	16
25	3979	3997	4014	4031	4048	4065	4082	4099	4116	4133	2	3	5	7	9	10	12	14	15
26	4150	4166	4183	4200	4216	4232	4249	4265	4281	4298	2	3	5	7	8	10	11	13	15
27	4314	4330	4346	4362	4378	4393	4409	4425	4440	4456	2	3	5	6	8	9	11	13	14
28	4472	4487	4502	4518	4533	4548	4564	4579	4594	4609	2	3	5	6	8	9	11	12	14
29	4624	4639	4654	4669	4683	4698	4713	4728	4742	4757	1	3	4	6	7	9	10	12	13
30	4771	4786	4800	4814	4829	4843	4857	4871	4886	4900	1	3	4	6	7	9	10	11	13
31	4914	4928	4942	4955	4969	4983	4997	5011	5024	5038	1	3	4	6	7	8	10	11	12
32	5051	5065	5079	5092	5105	5119	5132	5145	5159	5172	1	3	4	5	7	8	9	11	12
33	5185	5198	5211	5224	5237	5250	5263	5276	5289	5302	1	3	4	5	6	8	9	10	12
34	5315	5328	5340	5353	5366	5378	5391	5403	5416	5428	1	3	4	5	6	8	9	10	11
35	5441	5453	5465	5478	5490	5502	5515	5527	5539	5551	1	2	4	5	6	7	9	10	11
36	5563	5575	5587	5599	5611	5623	5635	5647	5658	5670	1	2	4	5	6	7	8	10	11
37	5682	5694	5705	5717	5729	5740	5752	5763	5775	5786	1	2	3	5	6	7	8	9	10
38	5798	5809	5821	5832	5843	5855	5866	5877	5888	5899	1	2	3	5	6	7	8	9	10
39	5911	5922	5933	5944	5955	5966	5977	5988	5999	6010	1	2	3	4	5	7	8	9	10
40	6021	6031	6042	6053	6064	6075	6085	6096	6107	6117	1	2	3	4	5	6	8	9	10
41	6128	6138	6149	6160	6170	6180	6191	6201	6212	6222	1	2	3	4	5	6	7	8	9
42	6232	6243	6253	6263	6274	6284	6294	6304	6314	6325	1	2	3	4	5	6	7	8	9
43	6335	6345	6355	6365	6375	6385	6395	6405	6415	6425	1	2	3	4	5	6	7	8	9
44	6435	6444	6454	6464	6474	6484	6493	6503	6513	6522	1	2	3	4	5	6	7	8	9
45	6532	6542	6551	6561	6571	6580	6590	6599	6609	6618	1	2	3	4	5	6	7	8	9
46	6628	6637	6646	6656	6665	6675	6684	6693	6702	6712	1	2	3	4	5	6	7	7	8
47	6721	6730	6739	6749	6758	6767	6776	6785	6794	6803	1	2	3	4	5	5	6	7	8
48	6812	6821	6830	6839	6848	6857	6866	6875	6884	6893	1	2	3	4	4	5	6	7	8
49	6902	6911	6920	6928	6937	6946	6955	6964	6972	6981	1	2	3	4	4	5	6	7	8
50	6990	6998	7007	7016	7024	7033	7042	7050	7059	7067	1	2	3	3	4	5	6	7	8
51	7076	7084	7093	7101	7110	7118	7126	7135	7143	7152	1	2	3	3	4	5	6	7	8
52	7160	7168	7177	7185	7193	7202	7210	7218	7226	7235	1	2	2	3	4	5	6	7	7
53	7243	7251	7259	7267	7275	7284	7292	7300	7308	7316	1	2	2	3	4	5	6	6	7
54	7324	7332	7340	7348	7356	7364	7372	7380	7388	7396	1	2	2	3	4	5	6	6	7

TABLE IV.

FOUR PLACE LOGARITHMS.

Natural numbers.	0	1	2	3	4	5	6	7	8	9	Proportional Parts. 1	2	3	4	5	6	7	8	9
55	7404	7412	7419	7427	7435	7443	7451	7459	7466	7474	1	2	2	3	4	5	5	6	7
56	7482	7490	7497	7505	7513	7520	7528	7536	7543	7551	1	2	2	3	4	5	5	6	7
57	7559	7566	7574	7582	7589	7597	7604	7612	7619	7627	1	2	2	3	4	5	5	6	7
58	7634	7642	7649	7657	7664	7672	7679	7686	7694	7701	1	1	2	3	4	4	5	6	7
59	7709	7716	7723	7731	7738	7745	7752	7760	7767	7774	1	1	2	3	4	4	5	6	7
60	7782	7789	7796	7803	7810	7818	7825	7832	7839	7846	1	1	2	3	4	4	5	6	6
61	7853	7860	7868	7875	7882	7889	7896	7903	7910	7917	1	1	2	3	4	4	5	6	6
62	7924	7931	7938	7945	7952	7959	7966	7973	7980	7987	1	1	2	3	3	4	5	6	6
63	7993	8000	8007	8014	8021	8028	8035	8041	8048	8055	1	1	2	3	3	4	5	5	6
64	8062	8069	8075	8082	8089	8096	8102	8109	8116	8122	1	1	2	3	3	4	5	5	6
65	8129	8136	8142	8149	8156	8162	8169	8176	8182	8189	1	1	2	3	3	4	5	5	6
66	8195	8202	8209	8215	8222	8228	8235	8241	8248	8254	1	1	2	3	3	4	5	5	6
67	8261	8267	8274	8280	8287	8293	8299	8306	8312	8319	1	1	2	3	3	4	5	5	6
68	8325	8331	8338	8344	8351	8357	8363	8370	8376	8382	1	1	2	3	3	4	4	5	6
69	8388	8395	8401	8407	8414	8420	8426	8432	8439	8445	1	1	2	2	3	4	4	5	6
70	8451	8457	8463	8470	8476	8482	8488	8494	8500	8506	1	1	2	2	3	4	4	5	6
71	8513	8519	8525	8531	8537	8543	8549	8555	8561	8567	1	1	2	2	3	4	4	5	5
72	8573	8579	8585	8591	8597	8603	8609	8615	8621	8627	1	1	2	2	3	4	2	5	5
73	8633	8639	8645	8651	8657	8663	8669	8675	8681	8686	1	1	2	2	3	4	4	5	5
74	8692	8698	8704	8710	8716	8722	8727	8733	8739	8745	1	1	2	2	3	4	4	5	5
75	8751	8756	8762	8768	8774	8779	8785	8791	8797	8802	1	1	2	2	3	3	4	5	5
76	8808	8814	8820	8825	8831	8837	8842	8848	8854	8859	1	1	2	2	3	3	4	5	5
77	8865	8871	8876	8882	8887	8893	8899	8904	8910	8915	1	1	2	2	3	3	4	4	5
78	8921	8927	8932	8938	8943	8949	8954	8960	8965	8971	1	1	2	2	3	3	4	4	5
79	8976	8982	8987	8993	8998	9004	9009	9015	9020	9026	1	1	2	2	3	3	4	4	5
80	9031	9036	9042	9047	9053	9058	9063	9069	9074	9079	1	1	2	2	3	3	4	4	5
81	9085	9090	9096	9101	9106	9112	9117	9122	9128	9133	1	1	2	2	3	3	4	4	5
82	9138	9143	9149	9154	9159	9165	9170	9175	9180	9186	1	1	2	2	3	3	4	4	5
83	9191	9196	9201	9206	9212	9217	9222	9227	9232	9238	1	1	2	2	3	3	4	4	5
84	9243	9248	9253	9258	9263	9269	9274	9279	9284	9289	1	1	2	2	3	3	4	4	5
85	9294	9299	9304	9309	9315	9320	9325	9330	9335	9340	1	1	2	2	3	3	4	4	5
86	9345	9350	9355	9360	9365	9370	9375	9380	9385	9390	1	1	2	2	3	3	4	4	5
87	9395	9400	9405	9410	9415	9420	9425	9430	9435	9440	0	1	1	2	2	3	3	4	4
88	9445	9450	9455	9460	9465	9469	9474	9479	9484	9489	0	1	1	2	2	3	3	4	4
89	9494	9499	9504	9509	9513	9518	9523	9528	9533	9538	0	1	1	2	2	3	3	4	4
90	9542	9547	9552	9557	9562	9566	9571	9576	9581	9586	0	1	1	2	2	3	3	4	4
91	9590	9595	9600	9605	9609	9614	9619	9624	9628	9633	0	1	1	2	2	3	3	4	4
92	9638	9643	9647	9652	9657	9661	9666	9671	9675	9680	0	1	1	2	2	3	3	4	4
93	9685	9689	9694	9699	9703	9708	9713	9717	9722	9727	0	1	1	2	2	3	3	4	4
94	9731	9736	9741	9745	9750	9754	9759	9763	9768	9773	0	1	1	2	2	3	3	4	4
95	9777	9782	9786	9791	9795	9800	9805	9809	9814	9818	0	1	1	2	2	3	3	4	4
96	9823	9827	9832	9836	9841	9845	9850	9854	9859	9863	0	1	1	2	2	3	3	4	4
97	9868	9872	9877	9881	9886	9890	9894	9899	9903	9908	0	1	1	2	2	3	3	4	4
98	9912	9917	9921	9926	9930	9934	9939	9943	9948	9952	0	1	1	2	2	3	3	4	4
99	9956	9961	9965	9969	9974	9978	9983	9987	9991	9996	0	1	1	2	2	3	3	3	4

TABLE V.

SINES, COSINES, TANGENTS.

	Natural.			Logarithmic.		
	Sin.	Cos.	Tan.	Sin.	Cos.	Tan.
0.0	0.0000	1.0000	0.0000	$-\infty$	0.0000	$-\infty$
0.5	0.0087	1.0000	0.0087	7.9408	0.0000	7.9409
1.	0.0175	0.9998	0.0175	8.2419	9.9999	8.2419
1.5	0.0262	0.9997	0.0262	8.4179	9.9999	8.4181
2.	0.0349	0.9994	0.0349	8.5428	9.9997	8.5431
2.5	0.0436	0.9990	0.0437	8.6397	9.9996	8.6401
3.	0.0523	0.9986	0.0524	8.7188	9.9994	8.7194
4.	0.0698	0.9976	0.0699	8.8436	9.9989	8.8446
5.	0.0872	0.9962	0.0875	8.9403	9.9983	8.9420
10.	0.1736	0.9848	0.1763	9.2397	9.9934	9.2463
15.	0.2588	0.9659	0.2679	9.4130	9.9849	9.4281
20.	0.3420	0.9397	0.3640	9.5341	9.9730	9.5611
25.	0.4226	0.9063	0.4663	9.6259	9.9573	9.6687
30.	0.5000	0.8660	0.5774	9.6990	9.9375	9.7614
35.	0.5736	0.8192	0.7002	9.7586	9.9134	9.8452
40.	0.6428	0.7660	0.8391	9.8081	9.8843	9.9238
45.	0.7071	0.7071	1.0000	9.8495	9.8495	0.0000
50.	0.7660	0.6428	1.1918	9.8843	9.8081	0.0762
55.	0.8192	0.5736	1.4281	9.9134	9.7586	0.1548
60.	0.8660	0.5000	1.7321	9.9375	9.6990	0.2386
65.	0.9063	0.4226	2.1445	9.9573	9.6259	0.3313
70.	0.9397	0.3420	2.7475	9.9730	9.5341	0.4389
75.	0.9659	0.2588	3.7321	9.9849	9.4130	0.5719
80.	0.9848	0.1736	5.6713	9.9934	9.2397	0.7537
85.	0.9962	0.0872	11.43	9.9983	8.9403	1.0580
90.	1.0000	0.0000	∞	0.0000	$-\infty$	∞